Combinatorics and graph theory 2.

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Hypergraphs: Erdős-Ko-Rado, Fischer, Sperner, LYM, Erdős-de Bruijn

Erdős-Ko-Rado theorem. (1961) $\mathcal{F} \subseteq 2^{[n]}$ k -uniform hypergraph ($k \le n/2$) with the property that if $A, B \in \mathcal{F}$-re $A \cap B \ne \emptyset$, then $|\mathcal{F}| \le \binom{n-1}{k-1}$ and this bound is sharp.

Fischer inequality. (1940) $\mathcal{F} = \{A_1, A_2, \dots, A_m\} \subseteq 2^{[n]}$ such that for every $i \ne j$ we have $|A_i \cap A_j| = \lambda > 0$. Then $m \le n$.

Ray-Chaudhuri-Wilson theorem. (1975) Let $L = \{l_1, l_2, \dots l_s\}$ and let $\mathcal{F} = \{A_1, A_2, \dots, A_m\} \subseteq 2^{[n]}$ such that for every $i \ne j$ we have $|A_i \cap A_j| \in L$. Thenwe have $m \le \sum_{i=0}^{s} \binom{n}{i}$.

Sperner theorem. (1928) $\mathcal{F} \subset 2^{[n]}$ and for any $A, B \in \mathcal{F}$, $A \not\subset B$ and $B \not\subset A$. Then $|\mathcal{F}| \le \binom{n}{\lfloor n/2 \rfloor}$. In case of equality $\mathcal{F}$ is exactly the family of all $\lfloor n/2 \rfloor$-subsets or all $\lceil n/2 \rceil$-subsets of $[n]$.

Erdős − De Bruijn theorem. (1948) $\mathcal{F} \subset 2^{[n]}$, for every $A \in \mathcal{F}$ $|A| \ge 2$, and for every $1 \le i < j \le n$ there is exactly one $A \in \mathcal{F}$ such that $i, j \in A$. Then $|\mathcal{F}| = 1$ or $|\mathcal{F}| \ge n$.

1. Let $n = p_1 p_2 \cdots p_k$, where every $p_i > 1$ and p_i prime. At most how many divisors of n can we take so that no two divisors are relative primes?
2. King Arthur sends n knights for reconnaissance trips. Every day k knights go. The same group can not go twice, and for safety reasons, any two groups have a common member. For how many days can he send out reconnaissance groups?
3. There are m lines on the plane, not all of them go through the same point and no two of them are parallel. Show that these lines determine at least m intersection points.
4. For a sophisticated agricultural experiment we need the following. We have m different plants, n different areas. On each area we have k different plants, each plant is on r different areas. Moreover, any two plants are together on l different areas, $l > r$. Prove that $n \ge m$.
5. We have some k-element sets, any two intersect in l elements. Prove that some element is contained in at most k sets.
6. Our base set has 100 elements. We take some 20- and some 80-element subsets such that any two subsets intersect. What is the maximum number of subsets with this condition?
7. The hypergraph $\mathcal{F} \subseteq 2^{[n]}$ has 2^{n-1} hyperedges and no two are disjoint. Show that there are $F_1, F_2 \in \mathcal{F}$ with exactly one common element.
8. We have a bipartite graph, the two classes are A and B. Any two vertices in A have exactly 97 common neighbors in B and any two vertices in B have exactly 111 common neighbors. a. Show such a graph. b. Show that there is no such graph with $|A| = |B| = 1000$.
9. Suppose that $k < n/2$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is an intersecting set system such that $|\mathcal{F}| = \binom{n-1}{k-1}$. Prove that there is an element in $[n]$, i, which is contained in all sets in $\mathcal{F}$. (The Erdős-Ko-Rado theorem, case of equality.)
10. Construct a set system $\mathcal{F} \subseteq 2^{[n]}$ such that any two sets in $\mathcal{F}$ intersect in at least two elements and $|\mathcal{F}| = 2^{n-2}$. Is there a larger set system with this property?

    Good to know: $\binom{2k}{k} \le 2^{2k-1}$. (Moreover, $\le 2^{2k}/\sqrt{k}$ if n is large enough.)
11. a. Let F be a tree of n vertices. At most how many *connected* subgraphs can we take such that none of them is a subgraph of another? b. At most how many *induced* subgraphs can we take such that none of them is a subgraph of another?
12. Let $\mathcal{F}$ be a set system which does not contain a chain of length $s+1$ (that is, sets $A_1 \subset A_2 \subset \dots \subset A_{s+1}$) Prove that $\sum_{k=0}^{n} \frac{f_k}{\binom{n}{k}} \le s$, where f_k is the number of sets of size k.

13. Let $\mathcal{F} \subseteq 2^{[2n]}$ be an intersecting set system such that all sets have an even number of elements. Prove that if n is also even then $\mathcal{F}$ contains at most 2^{2n-2} sets.

14. Suppose that any two edges of the hypergraph $\mathcal{H} = (V, \mathcal{E})$ are either disjoint or one contains the other. What is the maximum number of edges of $\mathcal{H}$?

15. At most how many clubs are possible in MOD-3-VILLAGE, which has n residents? Condition: if A_i is the set of members of the i-th club, then $|A_i| \not\equiv 0 \pmod 3$ and if $i \neq j$ then $|A_i \cap A_j| \equiv 0 \pmod 3$. (*)

16. Suppose that the hypergraph $\mathcal{H} = (V, \mathcal{E})$ does not contain a cycle. That is, there is no sequence of different vertices and edges $x_1, E_1, x_2, E_2, \ldots x_k, E_k, x_{k+1} = x_1$ such that E_i contains x_i and x_{i+1}. Suppose also that $\emptyset$ is not in $\mathcal{H}$, and that $\mathcal{H}$ is connected, that is, the vertex set V can not be divided into two nonempty parts, V_1 and V_2 such that all edges are in V_1 or V_2.

    Prove that in this case $\sum\{|E| - 1 : E \in \mathcal{E}\} = |V| - 1$.

17. Suppose that $k < n/3$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is a k-uniform set system with no three pairwise disjoint sets. Prove that $|\mathcal{F}| \leq 4\binom{n-1}{k-1}$.

Homework

1. Suppose that the set system $\mathcal{F} \subseteq 2^{[n]}$ has no two disjoint members. Show that there is an intersecting set system $\mathcal{F}' \subseteq 2^{[n]}$ which contains $\mathcal{F}$, and $|\mathcal{F}'| = 2^{n-1}$.

2. For any $k \geq 1$ show a k-uniform set system which is isomorphic to its dual.