ON A RESULT OF ANDRÁS BIRÓ

GERGELY HARCOS

We give a short proof of the following result of András Biró, originally proved in [2].

Theorem. *There exists* $0 < c < 1$ *with the following property. For every sufficiently large integer n there are n complex numbers* $z_1, \ldots, z_n$ *such that* $\max|z_i| = 1$ *and the n power sums*

$$s_\nu := z_1^\nu + \cdots + z_n^\nu, \qquad \nu = 1, \ldots, n,$$

are of modulus at most c.

Remark. The infimum of admissible c lies in $(0.5, 0.7)$. The upper bound follows from the proof below, while the lower bound was established in [1].

Proof. We can relax the condition $\max|z_i| = 1$ to $\max|z_i| \geqslant 1$. Let us use the notation

$$(z - z_1)\cdots(z - z_n) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n,$$

so that $a_0 = 1$. The condition $\max|z_i| \geqslant 1$ is certainly satisfied when $z = 1$ is a root of the left hand side, i.e. when

$$a_0 + \cdots + a_n = 0.$$

For $0 \leqslant m \leqslant n$ the inverse Newton–Girard formulae tell us that

$$a_m = \sum_{j_1 + 2j_2 + \cdots + nj_n = m} \frac{1}{j_1!\,j_2! \ldots j_n!} \left(\frac{-s_1}{1}\right)^{j_1} \left(\frac{-s_2}{2}\right)^{j_2} \cdots \left(\frac{-s_n}{n}\right)^{j_n},$$

where the j_ν's run through nonnegative integers, hence our task is to minimize $\max|s_\nu|$ under

$$\sum_{j_1 + 2j_2 + \cdots + nj_n \leqslant n} \frac{1}{j_1!\,j_2! \ldots j_n!} \left(\frac{-s_1}{1}\right)^{j_1} \left(\frac{-s_2}{2}\right)^{j_2} \cdots \left(\frac{-s_n}{n}\right)^{j_n} = 0.$$

If we write $T := \lfloor n/2 \rfloor$ and assume that

$$s_1 = \cdots = s_T = -w,$$

then in $s_{T+1}, \ldots, s_n$ the condition becomes linear:

$$\sum_{m<n/2} \left\{ \sum_{j_1 + 2j_2 + \cdots + mj_m \leqslant m} \frac{1}{j_1!\,j_2! \ldots j_n!} \left(\frac{w}{1}\right)^{j_1} \left(\frac{w}{2}\right)^{j_2} \cdots \left(\frac{w}{m}\right)^{j_m} \right\} \frac{s_{n-m}}{n-m}$$
$$= \sum_{j_1 + 2j_2 + \cdots + Tj_T \leqslant n} \frac{1}{j_1!\,j_2! \ldots j_n!} \left(\frac{w}{1}\right)^{j_1} \left(\frac{w}{2}\right)^{j_2} \cdots \left(\frac{w}{T}\right)^{j_T}.$$

The coefficients can be evaluated explicitly, so that the equation becomes

$$\sum_{m<n/2} \binom{w+m}{m} \frac{s_{n-m}}{n-m} = \binom{w+n}{n} - \sum_{m<n/2} \binom{w+m}{m} \frac{w}{n-m}.$$

After dividing by $\binom{w+n}{n}$ we obtain

$$\sum_{m<n/2} \prod_{\ell=m+1}^{n} \left(1+\frac{w}{\ell}\right)^{-1} \frac{s_{n-m}}{n-m} = 1 - \sum_{m<n/2} \prod_{\ell=m+1}^{n} \left(1+\frac{w}{\ell}\right)^{-1} \frac{w}{n-m},$$

which clearly has a solution satisfying

$$|s_{T+1}|,\dots,|s_n| \leqslant \left|1 - \sum_{m<n/2} \prod_{\ell=m+1}^{n} \left(1+\frac{w}{\ell}\right)^{-1} \frac{w}{n-m}\right| \Big/ \sum_{m<n/2} \left|\prod_{\ell=m+1}^{n} \left(1+\frac{w}{\ell}\right)^{-1} \frac{1}{n-m}\right|.$$

Here of course

$$\prod_{\ell=m+1}^{n} \left(1+\frac{w}{\ell}\right)^{-1} = \left(\frac{m}{n}\right)^{w} \left(1+O\left(\frac{1}{m}\right)\right),$$

hence a brief argument based on Riemann sums shows that under $n \to \infty$ the previous inequality becomes

$$|s_{T+1}|,\dots,|s_n| \leqslant o(1) + \left|1 - w\int_0^{1/2} \frac{t^w}{1-t}dt\right| \Big/ \int_0^{1/2} \frac{|t^w|}{1-t}dt.$$

Altogether we see that for any w in the unit disk, c is admissible as long as

$$\max\left\{|w|, \left|1 - w\int_0^{1/2} \frac{t^w}{1-t}dt\right| \Big/ \int_0^{1/2} \frac{|t^w|}{1-t}dt\right\} < c < 1.$$

For $w := -0.43246 - 0.54237i$ the left hand side is approximately 0.693676, hence

$$c := 0.693677$$

is admissible. □

REFERENCES

[1] A. Biró, *An improved estimate in a power sum problem of Turán*, Indag. Math. (N.S.) **11** (2000), 343–358.
[2] A. Biró, *An upper estimate in Turán's pure power sum problem*, Indag. Math. (N.S.) **11** (2000), 499–508.

ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS, HUNGARIAN ACADEMY OF SCIENCES, POB 127, BUDAPEST H-1364, HUNGARY
E-mail address: gharcos@renyi.hu