THE BOMBIERI–HALÁSZ–MONTGOMERY INEQUALITY

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We present two proofs of a useful generalization of Bessel's inequality. The statement is due to Bombieri [1] who attributes it to Selberg. In the context of zero density estimates, Bombieri's inequality is rooted in the work of Halász [3], Halász–Turán [4], and Montgomery [5]. Our exposition follows loosely Davenport [2, §27] and Montgomery [7, §5]. A more direct proof can be found in [6, §1], or in the original source [1].

Theorem 1 (Bombieri [1])**.** *Let ξ, $\phi_1, \ldots, \phi_R$ be vectors in a complex Hilbert space. Then*

$$\sum_r |\langle \xi, \phi_r\rangle|^2 \leqslant \|\xi\|^2 \max_s \sum_t |\langle \phi_s, \phi_t\rangle|. \tag{1}$$

First proof. Let us fix $\phi_1, \ldots, \phi_R$, and identify the smallest constant $B \geqslant 0$ such that

$$\sum_r |\langle \xi, \phi_r\rangle|^2 \leqslant B\|\xi\|^2 \tag{2}$$

holds for all ξ. Without loss of generality, ξ is of the form $\sum_s c_s\phi_s$ for some $(c_s) \in \mathbb{C}^R$. Then (2) says that

$$\sum_{s,t} c_s\overline{c_t} \sum_r \langle \phi_s, \phi_r\rangle\langle \phi_r, \phi_t\rangle \leqslant B \sum_{s,t} c_s\overline{c_t}\langle \phi_s, \phi_t\rangle$$

holds for all $(c_s) \in \mathbb{C}^R$. In other words, introducing the positive semidefinite matrix

$$A := (\langle \phi_s, \phi_t\rangle)_{1\leqslant s,t\leqslant R}, \tag{3}$$

the constant $B \geqslant 0$ is such that $BA - A^2$ is positive semidefinite. Diagonalizing A in an orthogonal basis of $\mathbb{C}^R$, we see that the smallest admissible B equals the largest eigenvalue $\rho(A)$ of A. Therefore, (1) is a consequence of the well-known bound [8, Prop. 7.6]

$$\rho(A) \leqslant \|A\|_\infty = \max_s \sum_t |\langle \phi_s, \phi_t\rangle|. \tag{4}$$

□

Second proof. As in the first proof, we shall identify the smallest constant $B \geqslant 0$ for (2). Let (ψ_n) be an orthonormal basis of the Hilbert space. Then we can write $\xi = \sum_n a_n\psi_n$, and (2) says that

$$\sum_r \left|\sum_n a_n\langle \psi_n, \phi_r\rangle\right|^2 \leqslant B\sum_n |a_n|^2, \qquad (a_n) \in \ell^2(\mathbb{C}).$$

Equivalently, by the duality principle (which is a consequence of the Cauchy–Schwartz inequality),

$$\sum_n \left|\sum_r c_r\langle \psi_n, \phi_r\rangle\right|^2 \leqslant B\sum_r |c_r|^2, \qquad (c_r) \in \mathbb{C}^R.$$

Replacing c_r by $\overline{c_r}$ and expanding the left-hand side, we obtain the alternative form

$$\sum_{s,t} c_s\overline{c_t}\langle \phi_s, \phi_t\rangle \leqslant B\sum_s |c_s|^2, \qquad (c_s) \in \mathbb{C}^R. \tag{5}$$

So with the notation (3), the matrix $B\cdot\mathrm{id} - A$ is positive semidefinite. As in the first proof, we conclude that the smallest admissible B equals $\rho(A)$, and then (1) follows by (4). □

Remark. In the second proof, we could have obtained (1) more directly, without recourse to eigenvalues. Indeed, we have

$$\sum_{s,t} c_s \overline{c_t} \langle \phi_s, \phi_t \rangle \leqslant \sum_{s,t} \frac{|c_s|^2 + |c_t|^2}{2} |\langle \phi_s, \phi_t \rangle| = \sum_s |c_s|^2 \sum_t |\langle \phi_s, \phi_t \rangle|,$$

whence (5) holds with

$$B = \max_s \sum_t |\langle \phi_s, \phi_t \rangle|.$$

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