ON THE MONOTONICITY OF DIRICHLET'S ETA FUNCTION

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We present David Speyer's elegant proof [3] that Dirichlet's eta function

$$\eta(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$$

is increasing on the positive axis $s > 0$. The statement was proved earlier by Jan van de Lune, see [2, Theorem 3].

The proof is based on an inequality essentially due to Chebyshev, cf. [1]:

Theorem 1. *Let $I \subseteq \mathbb{R}$ be an interval endowed with a finite measure m. Let $a, b \in L^1(I, m)$ be increasing integrable functions. Then*

$$\left(\int_I ab\,dm\right)\left(\int_I 1\,dm\right) \geqslant \left(\int_I a\,dm\right)\left(\int_I b\,dm\right).$$

Proof. Using the product measure $\mu := m \times m$ on $I \times I$, the inequality can be written as

$$\int_{(x,y)\in I^2} a(x)b(x)\,d\mu \geqslant \int_{(x,y)\in I^2} a(x)b(y)\,d\mu.$$

Doubling both sides, we can write the inequality in the more symmetric form

$$\int_{(x,y)\in I^2} a(x)b(x)+a(y)b(y)\,d\mu \geqslant \int_{(x,y)\in I^2} a(x)b(y)+a(y)b(x)\,d\mu.$$

Hence it suffices to prove that for any $x, y \in I$ we have

$$a(x)b(x)+a(y)b(y) \geqslant a(x)b(y)+a(y)b(x).$$

This, on the other hand, is clear from $(a(x)-a(y))(b(x)-b(y)) \geqslant 0$. □

Theorem 2. *Let $s \geqslant t > 0$. Then $\eta(s) \geqslant \eta(t)$.*

Proof. We apply the previous theorem with

$$I := (0,\infty), \qquad dm(x) := e^{-x}x^t\,\frac{dx}{x}, \qquad a(x) := \frac{1}{1+e^{-x}}, \qquad b(x) := x^{s-t}.$$

The conditions are clearly satisfied, whence

$$\left(\int_0^\infty \frac{e^{-x}}{1+e^{-x}}x^s\,\frac{dx}{x}\right)\left(\int_0^\infty e^{-x}x^t\,\frac{dx}{x}\right) \geqslant \left(\int_0^\infty \frac{e^{-x}}{1+e^{-x}}x^t\,\frac{dx}{x}\right)\left(\int_0^\infty e^{-x}x^s\,\frac{dx}{x}\right).$$

In other words,

$$\eta(s)\Gamma(s)\cdot\Gamma(t) \geqslant \eta(t)\Gamma(t)\cdot\Gamma(s),$$

and the statement follows. □

REFERENCES

[1] F. Franklin, *Proof of a theorem of Tchebycheff's on definite integrals*, Amer. J. Math. **7** (1885), 377–379, http://www.jstor.org/stable/2369183

[2] J. van de Lune, *Some inequalities involving Riemann's zeta-function*, Mathematisch Centrum, Afdeling Zuivere Wiskunde, ZW 50/75, Amsterdam, 1975, iii+15 pp., http://persistent-identifier.org/?identifier=urn:nbn:nl:ui:18-6895

[3] D. Speyer, Response to MathOverflow question No. 180716, http://mathoverflow.net/questions/180716

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