EULER MEETS FEYNMAN

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1. Some rational multiples of $\zeta(3)$

This note is based on the MathOverflow posts [1, 2, 3, 4] and the MathStackExchange post [5]. We shall evaluate some series and integrals by a combination of methods pioneered by Euler and Feynman. The values obtained are rational multiples of $\zeta(3)$.

Theorem 1. *Let H_n be the n-th harmonic number. Then*

$$\sum_{n=1}^{\infty}\frac{1}{n^2}H_n=2\zeta(3)\qquad \text{and} \qquad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}H_n=\frac{5}{8}\zeta(3).$$

Theorem 2.

$$\int_0^1\int_0^1\int_1^\infty\frac{dx\,dy\,dz}{x(x+y)(x+y+z)}=\frac{5}{24}\zeta(3).$$

Theorem 3.

$$\int_0^1\frac{\log(t)\log(t+1)}{t}\,dt=-\frac{3}{4}\zeta(3),$$
$$\int_0^1\frac{\log(t)\log(t+2)}{t+1}\,dt=-\frac{13}{24}\zeta(3),$$
$$\int_0^1\frac{\log(t)\log(t+1)}{t+1}\,dt=-\frac{1}{8}\zeta(3).$$

2. Proof of Theorem 1

Let $(a_n)_{n=1}^\infty$ be an arbitrary bounded sequence. Then

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{a_n}{n^2}H_n&=\sum_{n=1}^{\infty}\frac{a_n}{n^2}\left\{\frac{1}{n}+\frac{1}{2}\sum_{k=1}^{n-1}\left(\frac{1}{k}+\frac{1}{n-k}\right)\right\}\\&=\sum_{n=1}^{\infty}\frac{a_n}{n^3}+\frac{1}{2}\sum_{n=1}^{\infty}\sum_{k=1}^{n-1}\frac{a_n}{nk(n-k)}\\&=\sum_{n=1}^{\infty}\frac{a_n}{n^3}+\frac{1}{2}\sum_{k=1}^{\infty}\sum_{n=k+1}^{\infty}\frac{a_n}{nk(n-k)}\\&=\sum_{n\geqslant1}\frac{a_n}{n^3}+\frac{1}{2}\sum_{n,k\geqslant1}\frac{a_{n+k}}{nk(n+k)}.\end{aligned}$$

Comparing the two sides,

$$\sum_{n,k\geqslant1}\frac{a_{n+k}}{nk(n+k)}=2\sum_{n\geqslant1}\frac{a_n}{n^2}H_n-2\sum_{n\geqslant1}\frac{a_n}{n^3}. \tag{1}$$

On the other hand, we also have that

$$\sum_{n=1}^{\infty}\frac{a_n}{n^2}H_n=\sum_{n=1}^{\infty}\frac{a_n}{n^2}\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{n+k}\right)=\sum_{n,k\geqslant1}\frac{a_n}{nk(n+k)}.$$

Comparing the two sides, and using also the symmetry $n \leftrightarrow k$,

$$\sum_{n,k\geqslant 1}\frac{a_n}{nk(n+k)}=\sum_{n\geqslant 1}\frac{a_n}{n^2}H_n \qquad\text{and}\qquad \sum_{n,k\geqslant 1}\frac{a_k}{nk(n+k)}=\sum_{n\geqslant 1}\frac{a_n}{n^2}H_n. \tag{2}$$

If we take the constant sequence $a_n=1$, then (1) and (2) yield that

$$\sum_{n\geqslant 1}\frac{1}{n^2}H_n=\sum_{n,k\geqslant 1}\frac{1}{nk(n+k)}=2\zeta(3). \tag{3}$$

If we also utilize the parity sequence $a_n=(-1)^n$, then from (1) and (2) we deduce that

$$\sum_{n,k\geqslant 1}\frac{1+(-1)^n+(-1)^k+(-1)^{n+k}}{nk(n+k)}=2\sum_{n\geqslant 1}\frac{1+2(-1)^n}{n^2}H_n-2\sum_{n\geqslant 1}\frac{1+(-1)^n}{n^3}.$$

On the right-hand side, the numerator $1+(-1)^n$ equals 2 when n is even, and it equals 0 otherwise. Similarly, on the left-hand side, the numerator

$$1+(-1)^n+(-1)^k+(-1)^{n+k}=(1+(-1)^n)(1+(-1)^k)$$

equals 4 when n and k are even, and it equals 0 otherwise. Therefore, using also (3), we conclude that

$$\zeta(3)=4\zeta(3)+4\sum_{n\geqslant 1}\frac{(-1)^n}{n^2}H_n-\frac{1}{2}\zeta(3).$$

In other words,

$$\sum_{n\geqslant 1}\frac{(-1)^{n-1}}{n^2}H_n=\frac{5}{8}\zeta(3).$$

The proof of Theorem 1 is complete.

3. Proof of Theorem 2

Let us denote by I the integral to be evaluated. First we integrate with respect to x:

$$\begin{aligned}\int_1^\infty\frac{dx}{x(x+y)(x+y+z)}&=\left[\frac{\log(x)}{y(y+z)}+\frac{\log(x+y+z)}{(y+z)z}-\frac{\log(x+y)}{yz}\right]_{x=1}^{x=\infty}\\&=\frac{\log(1+y)}{yz}-\frac{\log(1+y+z)}{(y+z)z}.\end{aligned}$$

Now we integrate with respect to y, and make the change of variable $y\to t$ (resp. $y+z\to t$):

$$\begin{aligned}\int_0^1\int_1^\infty\frac{dx\,dy}{x(x+y)(x+y+z)}&=\int_0^1\frac{\log(1+t)}{tz}\,dt-\int_z^{1+z}\frac{\log(1+t)}{tz}\,dt\\&=\int_0^z\frac{\log(1+t)}{tz}\,dt-\int_1^{1+z}\frac{\log(1+t)}{tz}\,dt.\end{aligned}$$

Finally, we integrate with respect to z and apply Fubini's theorem:

$$\begin{aligned}I&=\int_0^1\int_0^z\frac{\log(1+t)}{tz}\,dt\,dz-\int_0^1\int_1^{1+z}\frac{\log(1+t)}{tz}\,dt\,dz\\&=\int_0^1\int_t^1\frac{\log(1+t)}{tz}\,dz\,dt-\int_1^2\int_{t-1}^1\frac{\log(1+t)}{tz}\,dz\,dt\\&=-\int_0^1\frac{\log(t)\log(t+1)}{t}\,dt+\int_1^2\frac{\log(t-1)\log(t+1)}{t}\,dt\\&=-\int_0^1\frac{\log(t)\log(t+1)}{t}\,dt+\int_0^1\frac{\log(t)\log(t+2)}{t+1}\,dt.\end{aligned}$$

We shall now obtain a similiar but different decomposition of I by utilizing the Feynman parametrization

$$\begin{aligned}\frac{1}{x(x+y)(x+y+z)} &= 2\int_0^1\int_0^{1-s}\frac{dr\,ds}{(rx+s(x+y)+(1-r-s)(x+y+z))^3}\\ &= 2\int_0^1\int_0^{1-s}\frac{dr\,ds}{(x+(1-r)y+(1-r-s)z)^3}\\ &= 2\int_0^1\int_0^{u}\frac{dv\,du}{(x+uy+vz)^3}.\end{aligned}$$

Integrating this with respect to x yields that

$$\int_1^\infty \frac{dx}{x(x+y)(x+y+z)} = \int_0^1\int_0^u \frac{dv\,du}{(1+uy+vz)^2}.$$

Integrating this with respect to y yields that

$$\int_0^1\int_1^\infty \frac{dx\,dy}{x(x+y)(x+y+z)} = \int_0^1\int_0^u \frac{dv\,du}{(1+vz)(1+u+vz)}.$$

Finally, integrating this with respect to z yields that the original integral equals

$$I = \int_0^1\int_0^u \frac{\log(1+u)+\log(1+v)-\log(1+u+v)}{uv}\,dv\,du.$$

Using that the integrand is symmetric in u and v, we infer that

$$I = \frac{1}{2}\int_0^1\int_0^1 \frac{\log(1+u)+\log(1+v)-\log(1+u+v)}{uv}\,du\,dv.$$

Now we integrate by parts in the u-variable to see that

$$\int_0^1 \frac{\log(1+u)+\log(1+v)-\log(1+u+v)}{uv}\,du = -\int_0^1 \frac{\log(u)}{(1+u)(1+u+v)}\,du.$$

Finally, we integrate with respect to v and change u to t to conclude that

$$I = \frac{1}{2}\int_0^1 \frac{\log(t)\log(t+1)}{t+1}\,dt - \frac{1}{2}\int_0^1 \frac{\log(t)\log(t+2)}{t+1}\,dt. \tag{4}$$

This should be compared with our earlier formula for I obtained by a different calculation:

$$I = -\int_0^1 \frac{\log(t)\log(t+1)}{t}\,dt + \int_0^1 \frac{\log(t)\log(t+2)}{t+1}\,dt. \tag{5}$$

Using (4) and (5), we can eliminate the difficult integral involving $\log(t+2)$:

$$3I = 2I + I = \int_0^1 \frac{\log(t)\log(t+1)}{t+1}\,dt - \int_0^1 \frac{\log(t)\log(t+1)}{t}\,dt.$$

The right-hand side equals

$$\begin{aligned}-\int_0^1 \frac{\log(t)\log(t+1)}{t(t+1)}\,dt &= \int_0^1 \frac{\log(t)}{t}\sum_{n=1}^\infty (-1)^n H_n t^n\,dt\\ &= \sum_{n=1}^\infty (-1)^n H_n \int_0^1 t^{n-1}\log(t)\,dt\\ &= \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^2}H_n = \frac{5}{8}\zeta(3),\end{aligned}$$

where in the last step we used Theorem 1. So this equals $3I$, and the proof of Theorem 2 is complete.

4. Proof of Theorem 3

The proof of the first identity is straightforward:

$$\begin{aligned}\int_0^1 \frac{\log(t)\log(t+1)}{t}\,dt &= \int_0^1 \frac{\log(t)}{t}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}t^n}{n}\,dt\\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^1 t^{n-1}\log(t)\,dt\\ &= \sum_{n=1}^{\infty}\frac{(-1)^n}{n^3} = -\frac{3}{4}\zeta(3).\end{aligned}$$

We combine this result with Theorem 2 and (5) to derive the second identity:

$$\int_0^1 \frac{\log(t)\log(t+2)}{t+1}\,dt = \frac{5}{24}\zeta(3) - \frac{3}{4}\zeta(3) = -\frac{13}{24}\zeta(3).$$

Then, we combine this result with Theorem 2 and (4) to derive the third identity:

$$\int_0^1 \frac{\log(t)\log(t+1)}{t+1}\,dt = \frac{5}{12}\zeta(3) - \frac{13}{24}\zeta(3) = -\frac{1}{8}\zeta(3).$$

The proof of Theorem 3 is complete.

References

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