ON SOME GEOMETRIC PROBABILITIES

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One can ask natural questions about a random triangle inscribed in the unit circle. By random we mean that the vertices are independently and uniformly distributed on the unit circle. We shall be concerned with the probability that the sides satisfy a modified triangle inequality. Two such questions were recently asked by Nilotpal Kanti Sinha with conjectured answers at MathStackExchange [4, 5]. The conjectures were confirmed at MathOverflow [2, 3] with the help of a clever method due to fedja at the site [1]. We present a simplified version of the method, and illustrate it with the two examples mentioned. The idea is to scale the random triangle so that one of the sides becomes 1, and then use the other two sides as parameters (instead of using the angles as parameters). This change of variables makes the calculations more tractable.

Let a, b, c be the sides of a random triangle inscribed in the unit circle. Without loss of generality, the vertices opposite these sides are $e^{2i\beta}$, $e^{-2i\alpha}$, 1, where $(\alpha,\beta)\in\mathbb{R}^2$ is uniformly distributed in the region

$$\alpha,\beta>0 \qquad \text{and} \qquad \alpha+\beta<\pi.$$

Then the triangle has angles α, β, $\pi-\alpha-\beta$, and

$$a=2\sin\alpha, \qquad b=2\sin\beta, \qquad c=2\sin(\alpha+\beta).$$

Let us consider

$$s:=\frac{b}{a}=\frac{\sin\beta}{\sin\alpha}, \qquad t:=\frac{c}{a}=\frac{\sin(\alpha+\beta)}{\sin\alpha}.$$

The mapping $(\alpha,\beta)\mapsto(s,t)$ is bijective onto the set of pairs $(s,t)\in\mathbb{R}^2$ satisfying

$$s,t>0 \qquad \text{and} \qquad s+t>1>|s-t|.$$

Moreover, the Jacobian of the mapping is st, whence

$$d\alpha\,d\beta=\frac{ds\,dt}{st}.$$

We note for later reference that the set of all pairs (α,β) has Lebesgue measure $\pi^2/2$, hence the subset corresponding to the event $c>b>a$ has Lebesgue measure $\pi^2/12$, and the subset corresponding to the event $b>a$ has Lebesgue measure $\pi^2/4$.

Theorem 1. *Assume that $p\geqslant 1$. Then*

$$P(a^p+b^p>c^p\mid c>b>a)=\frac{1}{p^2}. \tag{1}$$

Proof. Let $F(p)$ denote the conditional probability in question. Using the change of variables $(\alpha,\beta)\mapsto(s,t)$ described above, we see that

$$F(p)=\frac{12}{\pi^2}\int_{\Omega(p)}\frac{ds\,dt}{st},$$

where

$$\Omega(p):=\left\{(s,t)\in\mathbb{R}^2:(1+s^p)^{1/p}>t>s>1\right\}. \tag{2}$$

An application of Fubini's theorem shows that

$$F(p)=\frac{12}{\pi^2}\int_1^\infty\left(\int_s^{(1+s^p)^{1/p}}\frac{dt}{t}\right)\frac{ds}{s}=\frac{12}{\pi^2 p}\int_1^\infty\log(1+s^{-p})\frac{ds}{s}.$$

Expressing the last integral in terms of the new variable $u:=s^p$, we obtain that

$$F(p)=\frac{12}{\pi^2p^2}\int_1^\infty\log(1+u^{-1})\frac{du}{u}.$$

In particular,

$$F(p)=\frac{F(1)}{p^2}.$$

Since $F(1)=1$ by the triangle inequality, the result (1) follows. □

Theorem 2. *Assume that the pair* $(x,y)\in\mathbb{R}^2$ *satisfies* $x,y\leqslant 1$ *and* $x+y\geqslant 0$. *Then*

$$P(ax+by>c)=\frac{4}{\pi^2}\chi_2(x)+\frac{4}{\pi^2}\chi_2(y). \tag{3}$$

Proof. We shall calculate the conditional probability

$$F(x,y):=P(ax+by>c\mid b>a)$$

in terms of the dilogarithm function. The symmetry $a\leftrightarrow b$ and $x\leftrightarrow y$ reveals that

$$P(ax+by>c\mid a>b)=F(y,x),$$

hence in fact

$$P(ax+by>c)=\frac{F(x,y)+F(y,x)}{2}. \tag{4}$$

Using the change of variables $(\alpha,\beta)\mapsto(s,t)$ described above, we see that

$$G(x,y)=\frac{4}{\pi^2}\int_{\Omega(x,y)}\frac{ds\,dt}{st},$$

where

$$\Omega(x,y):=\left\{(s,t)\in\mathbb{R}^2: x+sy>t>s-1>0\right\}.$$

We note that the inequality $x+sy>t>s-1$ is void unless $\frac{1+x}{1-y}>s$; we interpret the fraction as ∞ when $y=1$. Furthermore, $\frac{1+x}{1-y}\geqslant 1$ holds by the initial conditions on x and y. Now an application of Fubini's theorem shows that

$$F(x,y)=\frac{4}{\pi^2}\int_1^{\frac{1+x}{1-y}}\left(\int_{s-1}^{x+sy}\frac{dt}{t}\right)\frac{ds}{s}=\frac{4}{\pi^2}\int_1^{\frac{1+x}{1-y}}\log\left(\frac{x+sy}{s-1}\right)\frac{ds}{s}.$$

We rewrite the last integral in terms of the new variable

$$u:=\frac{s-1}{x+sy}.$$

The result is that

$$F(x,y)=\frac{4}{\pi^2}\int_0^1-(\log u)\left(\frac{x}{1+ux}+\frac{y}{1-uy}\right)du.$$

However, for any $z\in[-1,1]$ we have that

$$\int_0^1-(\log u)\frac{z}{1-uz}\,du=-z\int_0^1(\log u)\sum_{n=0}^\infty(uz)^n=\sum_{n=0}^\infty\frac{z^{n+1}}{(n+1)^2}=\mathrm{Li}_2(z),$$

hence in fact

$$F(x,y)=\frac{4}{\pi^2}\mathrm{Li}_2(y)-\frac{4}{\pi^2}\mathrm{Li}_2(-x).$$

Finally, the result (3) follows from (4). □

REFERENCES

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