Overview

1. Spectral decomposition of $L^2(PSL_2(\mathbb{Z})\backslash\mathcal{H})$
   Maass forms, Eisenstein series
   Laplace operator, Hecke operators
   L-functions (modular, twisted, Rankin-Selberg)
   Extension to congruence subgroups

2. Riemann Hypothesis, Lindelöf Hypothesis
   The subconvexity problem
   A subconvex bound for twisted L-functions
   Amplification, shifted convolution sums
   Extension to modular and Rankin-Selberg L-functions

3. Heegner points on the modular surface
   Spectral theory and subconvexity for equidistribution
   Waldspurger like formulae

Spectral problem

$\mathcal{H} := \{ z = x + iy : y > 0 \}$ is a model of hyperbolic geometry (geodesics, $\frac{dx^2+dy^2}{y^2}$, $\frac{dx\,dy}{y^2}$, $PSL_2(\mathbb{R})$)

For a discrete subgroup $\Gamma \leq PSL_2(\mathbb{R})$ decompose $L^2(\Gamma \backslash \mathcal{H})$ into harmonics.

Scalar product: $\langle f, g \rangle := \int_{\Gamma\backslash\mathcal{H}} f \bar{g} \frac{dx\,dy}{y^2}$

In the theory of automorphic forms one usually restricts to Fuchsian groups of the first kind, i.e. to the case when $\Gamma\backslash\mathcal{H}$ has finite hyperbolic area. Then $\Gamma\backslash\mathcal{H}$ can be identified with a hyperbolic polygon with finitely many cusps on $\mathbb{R}\cup\{\infty\}$. For arithmetic the most important examples are $\Gamma = PSL_2(\mathbb{Z})$ and its congruence subgroups

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : N \mid c \right\}$$

I will focus mostly on $\Gamma = PSL_2(\mathbb{Z})$ and only indicate the new features for $N > 1$.

The Notion of harmonics

Let us consider the simpler examples $L^2(\mathbb{R})$ and $L^2(\mathbb{R}/\mathbb{Z})$:

- Any "nice" $f: \mathbb{R} \to \mathbb{C}$ can be decomposed as

$$f(x) = \int_{-\infty}^{\infty} \langle f, e^{2\pi i t \cdot} \rangle e^{2\pi i t x} \, dt$$

where $\langle f, e^{2\pi i t \cdot} \rangle = \int_{-\infty}^{\infty} f(x) e^{-2\pi i t x} \, dx$ is the Fourier tr.

- Any "nice" $f: \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ can be decomposed as

$$f(x) = \sum_{n=-\infty}^{\infty} \langle f, e^{2\pi i n \cdot} \rangle e^{2\pi i n x}$$

where $\langle f, e^{2\pi i n \cdot} \rangle = \int_0^1 f(x) e^{-2\pi i n x} \, dx$ is a Fourier coeff.

These extend to isometries $L^2(\mathbb{R}) \to L^2(\mathbb{R})$

$\int_{-\infty}^{\infty} f(x) \overline{g(x)} \, dx = \int_{-\infty}^{\infty} \langle f, e^{2\pi i t \cdot} \rangle \overline{\langle g, e^{2\pi i t \cdot} \rangle} \, dt$ $\quad L^2(\mathbb{R}/\mathbb{Z}) \to L^2(\mathbb{Z})$

$\int_0^1 f(x) \overline{g(x)} \, dx = \sum_{n=-\infty}^{\infty} \langle f, e^{2\pi i n \cdot} \rangle \overline{\langle g, e^{2\pi i n \cdot} \rangle}$

The harmonics are $x \mapsto e^{2\pi i t x}$, $x \mapsto e^{2\pi i n x}$ (respectively).

In the first case we have a continuous fam. of harmonics none of which lies in the L^2-space individually, the spectral measure is dt.

In the second case we have a discrete fam. of harmonics each of which lies in the L^2-space, the spectral measure is the counting meas.

The harmonics are precisely the eigenfunctions of the differential operator $\frac{d}{dx}$ which apparently generates the ring of invariant differential operators (wrt the action of $\mathbb{R}$)

In the case of $\mathcal{H}$ and $PSL_2(\mathbb{R})$, the ring of invariant differential operators is generated by the Laplacian operator $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$, which is positive and self-adjoint

<u>Theorem</u> (Selberg, 1958)

Let $\Gamma = PSL_2(\mathbb{Z})$. Then any nice $f: \Gamma\backslash\mathcal{H} \to \mathbb{C}$ has a decomposition into eigenfunctions of Δ:

$$f(z) = \sum_{j=0}^{\infty} \langle f, f_j \rangle \underbrace{f_j(z)}_{\text{in } L^2} + \int_{-\infty}^{\infty} \langle f, E(\cdot, \tfrac{1}{2}+it)\rangle \underbrace{E(z, \tfrac{1}{2}+it)}_{\text{not in } L^2} \frac{dt}{4\pi}$$

Here $f_0 = \text{const} = \sqrt{\frac{3}{\pi}}$, the f_j's ($j \geq 1$) are the Maass forms for Γ, $E(z, s)$ are Eisenstein series. If $\Delta f_j = \left(\frac{1}{4} + t_j^2\right) f_j$, then $t_j \in \mathbb{R}$, $\#\{t_j \leq T\} = \frac{1}{12}T^2 + O(T\log T)$. Also

$$\Delta E(z, \tfrac{1}{2}+it) = \left(\tfrac{1}{4} + t^2\right) E(z, \tfrac{1}{2}+it)$$

Hecke–Maass forms

It is not known if each t_j has multiplicity one (fails for some $N > 1$!), but with the help of Hecke operators one can further decompose each Δ-eigenspace and get down to multiplicity one eventually. The Hecke operators T_n form a commuting family of self-adjoint operators that also commutes with Δ. More precisely,

$$T_n f = \frac{1}{\sqrt{n}} \sum_{\tau \in \Gamma \backslash \Gamma_n} f(\tau z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \sum_{b \bmod d} f\left(\frac{az+b}{d}\right)$$

$$T_m T_n = \sum_{d \mid (m,n)} T_{\frac{mn}{d^2}} = T_n T_m \,, \quad T_n \Delta = \Delta T_n$$

$$\sum_{n=1}^{\infty} T_n n^{-s} = \prod_p \sum_{k=0}^{\infty} T_{p^k} p^{-ks} = \prod_p \left(1 - T_p p^{-s} + p^{-2s}\right)^{-1}$$

Usually one also considers $T_{-1} : f(x+iy) \mapsto f(-x+iy)$

Def. $f \in L^2(PSL_2(\mathbb{Z}) \backslash \mathcal{H})$ is a Hecke–Maass form if it is a simultaneous eigenfunction of Δ, T_p, T_{-1} for each prime p.

In Selberg's theorem we can choose the f_j's to be Hecke–Maass forms.

The eigenvalues of Δ, T_p, T_{-1} determine a 1-dimensional space. Indeed,

$$f_j(x+iy) = \sqrt{y} \sum_{n\neq 0} a_j(n) K_{it_j}(2\pi|n|y)\, e(nx)$$

where $a_j(-n) = \pm a_j(n)$ depending on the eigenvalue of T_{-1}

$$a_j(-n) = \lambda_j(-1) a_j(n)$$

$$\lambda_j(m) a_j(n) = \sum_{d|(m,n)} a_j\left(\frac{mn}{d^2}\right)$$

$$\boxed{a_j(\pm n) = \lambda_j(\pm 1) \lambda_j(n) a_j(1), \quad n \geq 1}$$

It also follows that $a_j(1) \neq 0$, so one can renormalize f_j to have $a_j(1) = 1$. This is the arithmetic normalization of a Maass form.

<u>Eisenstein series</u>

Let $\Gamma = PSL_2(\mathbb{Z})$, $\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \mid b \in \mathbb{Z} \right\}$,

$$E(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \operatorname{Im}(\gamma z)^s = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \frac{y^s}{|cz+d|^{2s}}.$$

Then $E(z,s)$ converges absolutely in $\operatorname{Re} s > 1$ and has a meromorphic continuation to $s \in \mathbb{C}$. In fact

$$E(z,s) = y^s + \phi(s) y^{1-s} + \frac{2\pi^s}{\Gamma(s)\zeta(2s)} \sqrt{y} \sum_{n\neq 0} \sigma_{1-2s}(n) |n|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|n|y)\, e(nx)$$

where

$$\phi(s) = \sqrt{\pi}\, \frac{\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\zeta(2s)}, \qquad \sigma_s(n) = \sum_{\substack{d|n \\ d>0}} d^s$$

From here it is easy to see that

$$E(z,s) = \phi(s)\, E(z, 1-s)$$

$$E(z, \tfrac{1}{2}) = 0 \qquad \text{because} \qquad \phi(\tfrac{1}{2}) = -1$$

~~And the only~~ In fact

$$\tilde{E}(z,s) := \tfrac{1}{2} \pi^{-s}\, \Gamma(s)\, \zeta(2s)\, E(z,s)$$ is symmetric

wrt $s \leftrightarrow 1-s$, the only poles are at $s = 0, 1$,

they are simple with residues $\pm \frac{1}{4}$

$$\tilde{E}(z,s) = \ldots y^{s} + \ldots y^{1-s} + \sqrt{y} \sum_{n \neq 0} \sigma_{1-2s}(|n|)\, |n|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|n|y)\, e(nx)$$

This is the arithmetic normalization:

The Eisenstein series are Hecke eigenforms

$$T_{-1}\, E(z,s) = E(z,s)$$

$$T_n\, E(z,s) = n^{s-\frac{1}{2}}\, \sigma_{1-2s}(n)\, E(z,s)$$

⇝ Project 1

In particular,

$$T_n\, E(z, \tfrac{1}{2}+it) = \Big(\sum_{ad=n} a^{it} d^{-it} \Big)\, E(z, \tfrac{1}{2}+it)$$

Also $\phi(s)\phi(1-s) = 1$, hence $|\phi(\frac{1}{2}+it)| = 1$,

so $E(z, \frac{1}{2} \pm it)$ are much the same.

Start Project 2 and Ramanujan Conjecture

Automorphic L-functions

Let f be a Hecke–Maass cusp form as above with Laplacian eigenvalue $\frac{1}{4}+t^2$. We associate to its Hecke eigenvalues $\lambda(n)$ the L-function

$$L(s,f) := \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \prod_p \frac{1}{1-\lambda(p)p^{-s}+p^{-2s}}.$$

More generally, if χ mod q is a primitive Dirichlet character, then we consider

$$L(s, f\times\chi) := \sum_{n=1}^{\infty} \frac{\lambda(n)\chi(n)}{n^s} = \prod_p \frac{1}{1-\lambda(p)\chi(p)p^{-s}+\chi(p)^2p^{-2s}}$$

Theorem Assume f, χ are both even. Then the expression

$$\tau(\bar\chi)\left(\frac{q}{\pi}\right)^s \Gamma\left(\frac{s+it}{2}\right)\Gamma\left(\frac{s-it}{2}\right) L(s, f\times\chi)$$

is holomorphic on $\mathbb{C}$, and is invariant under $\chi \to \bar\chi$, $s \to 1-s$.

Remark The proof works verbatim for f a newform of level dividing q.

Lemma Define $f_\chi := \sum^*_{r \bmod q} \bar{\chi}(r) f\left(z + \frac{r}{q}\right)$.

Then $f_\chi\left(\frac{-1}{q^2 z}\right) = f_{\bar{\chi}}(z)$.

Proof In slash notation we need to prove

$$f_\chi \Big| \begin{pmatrix} & -1 \\ q^2 & \end{pmatrix} = f_{\bar{\chi}} \qquad \text{i.e.}$$

$$\sum^*_{r \bmod q} \bar{\chi}(r) f\Big|\begin{pmatrix} 1 & r/q \\ & 1 \end{pmatrix}\begin{pmatrix} & -1 \\ q^2 & \end{pmatrix} = \sum^*_{s \bmod q} \chi(s) f\Big|\begin{pmatrix} 1 & s/q \\ & 1 \end{pmatrix}$$

It suffices to show that the r-term equals the $(-\bar{r})$-term on the right (where $r\bar{r} \equiv 1 \bmod q$), i.e.

$$f\Big|\begin{pmatrix} 1 & r/q \\ & 1 \end{pmatrix}\begin{pmatrix} & -1 \\ q^2 & \end{pmatrix} = f\Big|\begin{pmatrix} 1 & -\bar{r}/q \\ & 1 \end{pmatrix}, \quad \text{i.e.}$$

$$f\Big|\begin{pmatrix} 1 & r/q \\ & 1 \end{pmatrix}\begin{pmatrix} & -1 \\ q^2 & \end{pmatrix}\begin{pmatrix} 1 & \bar{r}/q \\ & 1 \end{pmatrix} = f$$

Here

$$\begin{pmatrix} 1 & r/q \\ & 1 \end{pmatrix}\begin{pmatrix} & -1 \\ q^2 & \end{pmatrix}\begin{pmatrix} 1 & \bar{r}/q \\ & 1 \end{pmatrix} = \begin{pmatrix} rq & -1 \\ q^2 & 0 \end{pmatrix}\begin{pmatrix} 1 & \bar{r}/q \\ & 1 \end{pmatrix}$$

$$= \begin{pmatrix} rq & r\bar{r}-1 \\ q^2 & \bar{r}q \end{pmatrix}$$

$$= \begin{pmatrix} q & \\ & q \end{pmatrix}\begin{pmatrix} r & \frac{r\bar{r}-1}{q} \\ q & \bar{r} \end{pmatrix}. \quad \square$$

Now consider, for $\mathrm{Re}\, s > 1$,

$$I(s, f\times\chi) := \int_0^\infty f\chi\left(\frac{iy}{q}\right) y^{s-\frac12}\frac{dy}{y}$$

$$= q^{s-\frac12}\int_0^\infty f\chi(iy)\, y^{s-\frac12}\frac{dy}{y} = q^{s-\frac12}\sum_{r \bmod q}^{*}\bar\chi(r)\int_0^\infty f\left(iy+\frac{r}{q}\right) y^{s-\frac12}\frac{dy}{y}$$

$$= q^{s-\frac12}\sum_{r \bmod q}^{*}\bar\chi(r)\sum_{n\neq 0} a(n)\, e\left(\frac{nr}{q}\right)\int_0^\infty K_{it}(2\pi|n|y)\, y^{s}\frac{dy}{y}$$

$$= q^{s-\frac12}\sum_{r \bmod q}^{*}\bar\chi(r)\sum_{n\neq 0}\frac{a(n)}{(2\pi|n|)^s} e\left(\frac{nr}{q}\right)\int_0^\infty K_{it}(y)\, y^{s}\frac{dy}{y}$$

$$= q^{s-\frac12}\sum_{r \bmod q}^{*}\bar\chi(r)\sum_{n\neq 0}\frac{a(n)}{(2\pi|n|)^s} e\left(\frac{nr}{q}\right) 2^{s-2}\,\Gamma\left(\frac{s+it}{2}\right)\Gamma\left(\frac{s-it}{2}\right)$$

$$= \frac14 q^{s-\frac12}\sum_{n\neq 0}\frac{a(n)}{(\pi|n|)^s}\left(\sum_{r \bmod q}^{*}\bar\chi(r)\, e\left(\frac{nr}{q}\right)\right)\Gamma\left(\frac{s+it}{2}\right)\Gamma\left(\frac{s-it}{2}\right)$$

$$= \frac{1}{4\sqrt q}\left(\frac{q}{\pi}\right)^s\sum_{n\neq 0}\frac{a(n)\chi(n)\tau(\bar\chi)}{|n|^s}\,\Gamma\left(\frac{s+it}{2}\right)\Gamma\left(\frac{s-it}{2}\right)$$

$$= \frac{\tau(\bar\chi)}{2\sqrt q}\Gamma\left(\frac{s+it}{2}\right)\Gamma\left(\frac{s-it}{2}\right)\left(\frac{q}{\pi}\right)^s\underbrace{\sum_{n=1}^\infty\frac{a(n)\chi(n)}{n^s}}_{a(1)L(s, f\times\chi)}$$

Convergence and other issues are taken care by the boundedness of $f\chi$ and Project 2.

On the other hand,

$$\underline{I}(s, f \times \chi) = \int_0^1 f\chi\left(\frac{iy}{q}\right) y^{s-\frac{1}{2}} \frac{dy}{y} + \int_1^\infty f\chi\left(\frac{iy}{q}\right) y^{s-\frac{1}{2}} \frac{dy}{y}$$

$$= \int_1^\infty \left\{ f\chi\left(\frac{i}{yq}\right) y^{\frac{1}{2}-s} + f\chi\left(\frac{iy}{q}\right) y^{s-\frac{1}{2}} \right\} \frac{dy}{y}$$

$$\overset{\text{Lemma}}{=} \int_1^\infty \left\{ f\bar{\chi}\left(\frac{iy}{q}\right) y^{\frac{1}{2}-s} + f\chi\left(\frac{iy}{q}\right) y^{s-\frac{1}{2}} \right\} \frac{dy}{y}$$

This is holomorphic and convergent for all $s \in \mathbb{C}$.

In addition,

$$\underline{I}(s, f \times \chi) = \underline{I}(1-s, f \times \bar{\chi}). \quad \square$$

The Theorem even holds when f is replaced by one of the Eisenstein series $E(\cdot, \frac{1}{2}+it)$ in the spectral decomposition. This is because

$$L(s, E(\cdot, \tfrac{1}{2}+it) \times \chi) = L(s+it, \chi) L(s-it, \chi) \rightsquigarrow \text{Project 1}$$

The subconvexity problem

Using the bound $\sum_{n \le N} |\lambda(n)|^2 \ll N$ from Project 2, it is straightforward to show

(1) $$L(s, f\times\chi) \ll_{\varepsilon, f} 1 \quad , \quad \mathrm{Re}(s) = 1+\varepsilon.$$

By the functional equation,

$$L(s, f\times\chi) = \frac{\tau(\chi)\left(\frac{q}{\pi}\right)^{1-s}\Gamma\left(\frac{1-s+it}{2}\right)\Gamma\left(\frac{1-s-it}{2}\right)L(1-s, f\times\bar{\chi})}{\tau(\bar{\chi})\left(\frac{q}{\pi}\right)^{s}\Gamma\left(\frac{s+it}{2}\right)\Gamma\left(\frac{s-it}{2}\right)}$$

$$\ll_{\sigma, f} q^{1-2\sigma}(1+|s|)^{1-2\sigma}\,|L(1-s, f\times\bar{\chi})| \quad , \quad \mathrm{Re}\, s = \sigma$$

Hence

(2) $$L(s, f\times\chi) \ll_{\varepsilon, f} (q|s|)^{1+\varepsilon} \quad , \quad \mathrm{Re}\, s = -\varepsilon$$

We can interpolate between (1) and (2) using the Phragmén-Lindelöf convexity principle, and obtain the so-called convexity bound in the critical strip:

$$L(s, f\times\chi) \ll_{\varepsilon, f, \sigma} (q|s|)^{1-\sigma+\varepsilon} \quad , \quad \mathrm{Re}\, s = \sigma$$

Subconvexity problem (χ-aspect):

Find some $\delta > 0$ and $A > 0$ s.t.

(3) $L(s, f \times \chi) \ll_{f,\varepsilon} q^{\frac{1}{2}-\delta+\varepsilon} |s|^A$, $\operatorname{Re} s = \frac{1}{2}$.

By standard techniques involving the Mellin transform, we can reformulate this to a statement about cancellation in finite sums:

(4) $\sum_{n=1}^{\infty} \lambda(n)\chi(n)V(n) \ll_{f,\varepsilon} X^{\frac{1}{2}} q^{\frac{1}{2}-\delta+\varepsilon} Z^B$

for any smooth $V : (0,\infty) \to \mathbb{C}$ supported in $[X, 2X]$ s.t. $V^{(j)} \ll (Z/X)^j$ for any $j \geq 0$.

We can assume $X \ll_{f,\varepsilon} q^{1+\varepsilon}$ here.

Riemann Hypothesis

All the zeros of $\Gamma(\frac{s+it}{2})\Gamma(\frac{s-it}{2}) L(s, f \times \chi)$ lie on $\operatorname{Re} s = \frac{1}{2}$

Lindelöf Hypothesis

$\delta = \frac{1}{2}$ is valid in (3) and (4)

- $\delta = \frac{1}{22}$ by Duke-Friedlander-Iwaniec (1993)
- $\delta = \frac{1}{8}$ by Bykovskii (1996), Blomer-Harcos (2008)
- $\delta = \frac{1}{6}$ for quadratic χ and $s = \frac{1}{2}$ by Conrey-Iwaniec (2000).

These results are also valid when $f(z) = E(z, \frac{1}{2}+it)$ due to Burgess (1963) and Conrey–Iwaniec (2000).

To establish (4) we consider the amplified second mean

$$S := \sum_{w \in \widehat{(\mathbb{Z}/q)^\times}} \Big| \sum_{\ell \sim L} \bar{\chi}(\ell) w(\ell) \Big|^2 |S_w|^2 .$$

where $S_w := \sum_{n \sim X} \lambda(n) w(n) v(n)$.

Remember that we want

$$S_\chi \ll_{f,\varepsilon,v} X^{\frac{1}{2}} q^{\frac{1}{2}-\delta+\varepsilon}$$

for $X \ll_{f,\varepsilon} q^{1+\varepsilon}$

$$S = \sum_{w \in \widehat{(\mathbb{Z}/q)^\times}} \Big| \sum_{\substack{\ell \sim L \\ n \sim X}} \bar{\chi}(\ell) \lambda(n) w(\ell n) v(n) \Big|^2$$

$$= \sum_{w \in \widehat{(\mathbb{Z}/q)^\times}} \Big| \sum_{m \in (\mathbb{Z}/q)^\times} w(m) \sum_{\substack{\ell \sim L \\ n \sim X \\ \ell n \equiv m\,(q)}} \bar{\chi}(\ell) \lambda(n) v(n) \Big|^2 .$$

By Plancherel for $(\mathbb{Z}/q)^\times$,

$$S = \varphi(q) \sum_{m \in (\mathbb{Z}/q)^\times} \Big| \sum_{\substack{l \sim L \\ n \sim X \\ ln \equiv m\,(q)}} \bar{\chi}(l)\,\lambda(n)\,V(n) \Big|^2$$

$$\leq q \sum_{m \in \mathbb{Z}/q} \Big| \sum_{\substack{l \sim L \\ n \sim X \\ ln \equiv m\,(q)}} \bar{\chi}(l)\,\lambda(n)\,V(n) \Big|^2$$

$$= q \sum_{\substack{l_1, l_2 \sim L \\ n_1, n_2 \sim X \\ l_1 n_1 \equiv l_2 n_2\,(q)}} \bar{\chi}(l_1)\,\chi(l_2)\,\lambda(n_1)\,\overline{\lambda(n_2)}\,V(n_1)\,\overline{V(n_2)}$$

Hence, denoting $h := l_1 n_1 - l_2 n_2$,

$$q^{-1} S \ll \underbrace{\Big| \sum_{\substack{l_1 l_2 \sim L \\ n_1, n_2 \sim X \\ l_1 n_1 = l_2 n_2}} \lambda(n_1)\,\overline{\lambda(n_2)} \Big|}_{\text{diagonal}} + \underbrace{\sum_{\substack{1 \leq h \leq 4LX \\ h \equiv 0\,(q)}} \sum_{l_1 l_2 \sim L} |D(l_1, l_2, h)|}_{\text{off-diagonal}}$$

where

$$D(l_1, l_2, h) := \sum_{\substack{n_1, n_2 \sim X \\ l_1 n_1 - l_2 n_2 = h}} \lambda(n_1)\,\overline{\lambda(n_2)}\,V(n_1)\,\overline{V(n_2)}$$

is a shifted convolution sum

The diagonal term can be estimated by Cauchy-Schwarz:

$$\left| \sum_{\substack{l_1, l_2 \sim L \\ n_1, n_2 \sim X \\ l_1 n_1 = l_2 n_2}} \lambda(n_1) \overline{\lambda(n_2)} \right| \leq \frac{1}{2} \sum_{\substack{l_1, l_2 \sim L \\ n_1, n_2 \sim X \\ l_1 n_1 = l_2 n_2}} \left(|\lambda(n_1)|^2 + |\lambda(n_2)|^2 \right)$$

$$= \sum_{\substack{l \sim L \\ n \sim X}} |\lambda(n)|^2 \tau(ln) \ll_{f,\varepsilon} q^\varepsilon L X$$

<u>Assumption</u> For $1 \leq h \leq 4LX$ and $l_1, l_2 \sim L$ we have $D(l_1, l_2, h) \ll_{f,\nu,\varepsilon} X^{1-\delta+\varepsilon}$

Then

$$q^{-1-\varepsilon} S \ll_{f,\varepsilon,\nu} LX + \frac{LX}{q} L^2 X^{1-\delta}$$

Assuming $L \geq q^{2\varepsilon}$ we have

$$S \geq \Big| \sum_{\substack{l \sim L \\ (l,q)=1}} 1 \Big|^2 |S_\chi|^2 \gg_\varepsilon q L^2 |S_\chi|^2, \text{ so that}$$

$$q^{-\varepsilon} |S_\chi|^2 \ll_{f,\varepsilon,\nu} q \frac{X}{L} + L X^{2-\delta}$$

Optimal bound when

$$\varrho \frac{X}{L} = L X^{2-\delta}, \quad L = \varrho^{\frac{1}{2}} X^{-\frac{1}{2}+\frac{\delta}{2}} \quad \left(> \varrho^{\frac{\delta}{2}-\varepsilon} > \varrho^{\frac{\delta}{3}} \right)$$

yielding

$$|S_X|^2 \ll_{f,\varepsilon,\nu} \varrho^{\frac{1}{2}+\varepsilon} X^{\frac{3}{2}-\frac{\delta}{2}} \ll_{f,\varepsilon,\nu} \varrho^{1-\frac{\delta}{2}+\varepsilon} X$$

$$S_X \ll_{f,\varepsilon,\nu} \varrho^{\frac{1}{2}-\frac{\delta}{4}+\varepsilon} X^{\frac{1}{2}}$$

Blomer (2004) established $D(\ell_1,\ell_2,h) \ll_{f,\nu,\varepsilon} (LX)^{\frac{1}{2}+\theta+\varepsilon}$

where $\theta = \frac{7}{64}$. This yields

$$\varrho^{-\varepsilon} |S_X|^2 \ll_{f,\varepsilon,\nu} \varrho \frac{X}{L} + (LX)^{\frac{3}{2}+\theta}$$

Optimal bound when $L = \varrho^{\frac{2}{5+2\theta}} X^{-\frac{1+2\theta}{5+2\theta}}$ yielding

$$\varrho^{-\varepsilon} |S_X|^2 \ll_{f,\varepsilon,\nu} \varrho^{\frac{3+2\theta}{5+2\theta}} X^{\frac{6+4\theta}{5+2\theta}} \ll_{f,\varepsilon,\nu} \varrho^{\varepsilon+\frac{4+4\theta}{5+2\theta}} X$$

$$S_X \ll_{f,\varepsilon,\nu} \varrho^{\frac{1}{2}-\frac{1-2\theta}{10+4\theta}+\varepsilon} X^{\frac{1}{2}}$$

Further progress by averaging over h, or amplifying over f (not X)

B7

By spectral methods one can refine Blomer's bound to

$$D(\ell_1,\ell_2,h) \ll_{f,\nu,\varepsilon} (LX)^{\frac{1}{2}} \text{ "spectral average of } \lambda(h) \text{ over } \Gamma_0(\ell_1\ell_2)\backslash\mathcal{H}\text{"}$$

which yields, uniformly in $\ell_1, \ell_2 \sim L$,

$$\sum_{\substack{1\leq h\leq 4LX \\ h\equiv 0(q)}} D(\ell_1,\ell_2,h) \ll_{f,\nu,\varepsilon} (LX)^{\frac{1}{2}} q^{\theta+\varepsilon}\left(\frac{LX}{q}\right)^{\frac{1}{2}} = q^{-\frac{1}{2}+\theta+\varepsilon} LX$$

$$q^{-\varepsilon}|S_X|^2 \ll_{f,\nu,\varepsilon} q\frac{X}{L} + q^{+\frac{1}{2}+\theta} LX.$$

Optimal bound when $L = q^{\frac{1-2\theta}{4}}$, yielding

$$S_X \ll_{f,\nu,\varepsilon} q^{\frac{1}{2}-\frac{1-2\theta}{8}+\varepsilon} X^{\frac{1}{2}}.$$

For details see Blomer - Harcos - Michel (Forum Math. 2007)

. Blomer - Harcos (GAFA 2010).

Equidistribution on the modular surface

The subconvex bound

$$L(s, f\times\chi) \ll_{f,\varepsilon} q^{\frac{1}{2}-\delta+\varepsilon} |s|^A \quad , \operatorname{Re} s = \frac{1}{2}$$

can be applied to equidistribution problems on hyperbolic surfaces and ellipsoids. I will focus on the distribution of Heegner points on the modular surface, corresponding to a negative fundamental discriminant d. We shall only need the bound for $s=\frac{1}{2}$, $\chi = \left(\frac{d}{\cdot}\right)$, but in the more precise form

$$L\left(\tfrac{1}{2}, f\times\left(\tfrac{d}{\cdot}\right)\right) \ll |d|^{\frac{1}{2}-\delta+\varepsilon}(1+|t|)^B$$

where $\Delta f = \left(\frac{1}{4}+t^2\right)f$.

Recall that a fundamental discriminant d is such that every integral binary quadratic form $ax^2+bxy+cy^2$ with that discriminant has coprime coefficients. These are the numbers "square-free congruent to 1 mod 4" and "4 times square-free congruent to 2, 3 mod 4"

Example $d = -3, -4, -7, -8, -11, -15, -19, -20, -23, -24$

$d = 5, 8, 12, 13, 17, 21, 24, 28, 29, 33$

C2

The modular group $SL_2(\mathbb{Z})$ acts on the set of integral binary quadratic forms with a given discriminant d:

$$ax^2+bxy+cy^2 \xmapsto{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}} a(\alpha x+\beta y)^2+b(\alpha x+\beta y)(\gamma x+\delta y)+c(\gamma x+\delta y)^2$$

Lagrange (1773) discovered that the number of orbits is finite. We can prove this fact by associating to each orbit a so-called Heegner point on $SL_2(\mathbb{Z})\backslash\mathcal{H}$. Namely, we assign to $ax^2+bxy+cy^2$ the point $z\in\mathcal{H}$ s.t.

$$ax^2+bxy+cy^2 = a(x-zy)(x-\bar{z}y) \quad \text{i.e.}$$

$$z=\frac{-b+\sqrt{d}}{2a}, \quad \bar{z}=\frac{-b-\sqrt{d}}{2a} \quad \left(z+\bar{z}=-\frac{b}{a},\ z\bar{z}=\frac{c}{a}\right)$$

Each orbit results in a unique point in the standard fundamental domain for $SL_2(\mathbb{Z})\backslash\mathcal{H}$. What does it mean that z lies in the closure?

$$|\mathrm{Re}\, z| \leq \tfrac{1}{2} \iff |b|\leq |a|$$

$$|z|\geq 1 \iff |a|\leq |c|.$$

So the number of orbits (or equivalence classes) is at most the number of triplets (a,b,c) s.t. $|b|\leq|a|\leq|c|$ and $b^2-4ac=d$.

Now we have

$$|d| = 4ac - b^2 \geq 3b^2 \quad , \quad \text{hence there}$$

are $O(|d|^{\frac{1}{2}})$ choices for b, so $O(|d|^{\frac{1}{2}+\varepsilon})$ choices for (a,b,c). This bound is essentially sharp:

<u>Theorem</u> (Dirichlet + Siegel)

$$|d|^{\frac{1}{2}-\varepsilon} \ll_\varepsilon h(d) \ll_\varepsilon |d|^{\frac{1}{2}+\varepsilon}.$$

~~Example The equivalence classes for $d=-23$ are represented by $x^2+xy+6y^2$, $2x^2+xy+3y^2$, $2x^2-xy+3y^2$. Hence there are 3 Heegner points of discriminant -23 on $SL_2(\mathbb{Z})\backslash\mathcal{H}$.~~

<u>Question</u> Given a very small d, how ~~are~~ is the associated set of Heegner points $\Lambda_d \subset SL_2(\mathbb{Z})\backslash\mathcal{H}$ distributed? / We would like to see equidistribution, i.e.

$$\frac{1}{h(d)} \sum_{z \in \Lambda_d} f(z) \longrightarrow \int_{SL_2(\mathbb{Z})\backslash\mathcal{H}} f(z) \frac{3}{\pi} \frac{dx\,dy}{y^2} \quad , \quad d \to -\infty$$

for any nice $f: SL_2(\mathbb{Z})\backslash\mathcal{H} \to \mathbb{C}$ (e.g. smooth, compactly supp)

C4

We have $\frac{1}{h(d)} \sum_{z \in \Lambda_d} f(z)$ Call the left hand side $\omega_d(f)$.

Call the left hand side $S_d(f)$. Then the spectral expansion

$$f(z) = \sum_{j \geq 0} \langle f, f_j \rangle f_j(z) + \int_{-\infty}^{\infty} \langle f, E(\cdot, \tfrac{1}{2}+it) \rangle E(z, \tfrac{1}{2}+it) \frac{dt}{4\pi}$$

yields

$$S_d(f) = \sum_{j \geq 0} \langle f, f_j \rangle S_d(f_j) + \int_{-\infty}^{\infty} \langle f, E(\cdot, \tfrac{1}{2}+it) \rangle S_d(E(\cdot, \tfrac{1}{2}+it)) \frac{dt}{4\pi}$$

Convergence issues are taken care by

$$(1+t_j^2)^c \langle f, f_j \rangle = \langle f, \Delta^c f_j \rangle = \langle \Delta^c f, f_j \rangle \ll_{f,c} 1$$

$$(1) \begin{cases} \text{i.e. } \langle f, f_j \rangle \ll_{f,c} (1+|t_j|)^{-2c} \quad \text{and similarly} \\ \langle f, E(\cdot, \tfrac{1}{2}+it) \rangle \ll_{f,c} (1+|t|)^{-2c}. \end{cases}$$

Observe that

$$\langle f, f_0 \rangle S_d(f_0) = \frac{3}{\pi} \langle f, 1 \rangle = \int_{SL_2(\mathbb{Z}) \backslash \mathcal{H}} f(z) \frac{3}{\pi} \frac{dx\,dy}{y^2}$$

so we want to see that

$$S_d(f) - \langle f, f_0\rangle S_d(f_0) = \sum_{j\geq 1} \langle f, f_j\rangle S_d(f_j) + \dots$$

is small. By (1) and the Weyl law it suffices to prove

$$S_d(f_j) \ll (1+|t_j|)^c |d|^{-\delta}, \quad S_d(E(\cdot, \tfrac{1}{2}+it)) \ll (1+|t|)^c |d|^{-\delta}$$

for some $c > 0$, $\delta > 0$. These ~~expressio~~ sums can be expressed in terms of twisted L-functions:

<u>Theorem</u> (Dirichlet, Hecke, Siegel, Maass, Shimura, Waldspurger, Kohnen-Zagier, Duke, Katok-Sarnak, Guo)

$$\left| \sum_{z \in \Lambda_d} f_j(z) \right|^2 = c_d |d|^{\frac{1}{2}} |a_j(1)|^2 \Lambda(\tfrac{1}{2}, f_j) \Lambda(\tfrac{1}{2}, f_j \otimes (\tfrac{d}{\cdot}))$$

and similarly for $E(z, \frac{1}{2}+it)$ in place of $f_j(z)$.

Here $a_j(1)$ is the first Fourier coefficient. Its growth is well-understood (Iwaniec, Hoffstein-Lockhart) and is cancelled by the exponential decay of the gamma-factors in $\Lambda(\dots)$. Noting also the lower bound for $h(d)$, we obtain

• ~~$\sum_{z\in\Lambda_d} f_j(z)$~~

$$|S_d(f_j)|^2 \ll (1+|t_j|)^2 |d|^{-\frac{1}{2}+\varepsilon} L\left(\tfrac{1}{2}, f_j \otimes \left(\tfrac{d}{\cdot}\right)\right)$$

Therefore a subconvex bound $L\left(\frac{1}{2}, f_j \otimes \left(\frac{d}{\cdot}\right)\right) \ll (1+|t_j|)^B |d|^{\frac{1}{2}-\delta+\varepsilon}$ yields $S_d(f_j) \ll (1+|t_j|)^{1+\frac{B}{2}} |d|^{-\frac{\delta}{2}+\varepsilon}$ and finally

$$\frac{1}{h(d)} \sum_{z\in\Lambda_d} f(z) = \int_{SL_2(\mathbb{Z})\backslash\mathcal{H}} f(z) \frac{3}{\pi} \frac{dx\,dy}{y^2} + O_f\left(|d|^{-\frac{\delta}{2}+\varepsilon}\right).$$

This result was conjectured by Linnik (1968) and proved by Duke (1988) with exponent $-\frac{1}{28}$ (corresponding to $\delta = \frac{1}{14}$). The Burgess exponent yields $-\frac{1}{16}$ and the Conrey–Iwaniec bound yields $-\frac{1}{12}$.

Let us evaluate $\left|\sum_{z\in\Lambda_d} E(z,\tfrac{1}{2}+it)\right|^2$ C7

For $\operatorname{Re} s>1$ we have

$$\sum_{z\in\Lambda_d} E(z,s) = \frac{1}{2}\sum_{z\in\Lambda_d}\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)=1}} \frac{(|d|^{\frac{1}{2}}/2a)^s}{|mz+n|^s}$$

$$= |d|^{\frac{s}{2}} 2^{-s-1} \sum_{\langle a,b,c\rangle\in Q_d} \sum_{\substack{(m,n)\in\mathbb{Z}\\(m,n)=1}} \frac{1}{(am^2+bmn+cn^2)^s}$$

$$= w\frac{|d|^{\frac{s}{2}} 2^{-s-1}}{\zeta(2s)} \sum_{[I]\in C_d} \sum_{\alpha\in I} \frac{N(I)^s}{N(\alpha)^s}$$

$$= w\frac{|d|^{\frac{s}{2}} 2^{-s-1}}{\zeta(2s)} \sum_{\substack{I\trianglelefteq O_d\\ I\neq 0}} \frac{1}{N(I)^s} = w\frac{|d|^{\frac{s}{2}}}{2^2(2\pi)^s}\Gamma(s)\, a_s(1)\, \zeta_{\mathbb{Q}(\sqrt{-d})}(s)$$

since $a_s(1) = \dfrac{2\pi^s}{\Gamma(s)\zeta(2s)}$ for $E(z,s)$. Now

$$\frac{\Gamma(s)}{2^2(2\pi)^s}\zeta_{\mathbb{Q}(\sqrt{-d})}(s) = \frac{1}{2^2(2\pi)^s}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)2^{s-1}\pi^{-\frac{1}{2}}\zeta(s)L\left(s,\left(\frac{d}{\cdot}\right)\right)$$

$$= \frac{1}{8}\left\{\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)\right\}\left\{\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)L\left(s,\left(\frac{d}{\cdot}\right)\right)\right\}$$

$$= \frac{1}{8}\Lambda(s,1)\Lambda\left(s,\left(\frac{d}{\cdot}\right)\right),$$ hence

C8

$$\sum_{z\in\Lambda_d} E(z,s) = \frac{w|d|^{\frac{s}{2}}}{8}\, a_s(1)\,\Lambda(s,1)\,\Lambda\left(s,\left(\tfrac{d}{\cdot}\right)\right)$$

so by meromorphic continuation

$$\left|\sum_{z\in\Lambda_d} E\left(z,\tfrac12+it\right)\right|^2 = \frac{w^2|d|^{\frac12}}{64}\left|a_{\frac12+it}(1)\right|^2 \left|\Lambda\left(\tfrac12+it,1\right)\right|^2 \left|\Lambda\left(\tfrac12+it,\left(\tfrac{d}{\cdot}\right)\right)\right|^2$$

$$= \frac{w^2|d|^{\frac12}}{64}\left|a_{\frac12+it}(1)\right|^2 \Lambda\left(\tfrac12, E\left(\cdot,\tfrac12+it\right)\right)\Lambda\left(\tfrac12, E\left(\cdot,\tfrac12+it\right)\otimes\left(\tfrac{d}{\cdot}\right)\right)$$

Project 1

(1) $E(z,s)$ converges absolutely for $\mathrm{Re}\, s > 1$

(2) $\tilde{E}(z,s) = \frac{1}{2}\pi^{-s}\Gamma(s)\zeta(2s)E(z,s)$ is meromorphic in s with the only poles at $s = 0, 1$ which are simple

(3) $\tilde{E}(z,s) = \tilde{E}(z, 1-s)$

(4) $T_n E(z,s) = n^{s-\frac{1}{2}}\sigma_{1-2s}(n)E(z,s)$

(5) $L(s, E(\cdot, \tfrac{1}{2}+it) \times \chi) = L(s+it, \chi)L(s-it, \chi)$

Project 2

Let $f(x+iy) = \sqrt{y}\sum_{n\neq 0} a(n)K_{it}(2\pi|n|y)e(nx)$ be a Maass form for $SL_2(\mathbb{Z})$. Show that

$$\sum_{|n|\leq N} |a(n)|^2 \ll_f N$$

Deduce that f decays exponentially in the cusp.

Project 3

Let

$$f(x+iy) = \sqrt{y}\sum_{n\neq 0} a(n)K_{it_f}(2\pi|n|y)e(nx)$$

$$g(x+iy) = \sqrt{y}\sum_{n\neq 0} b(n)K_{it_g}(2\pi|n|y)e(nx)$$

be two even Maass forms for $SL_2(\mathbb{Z})$. Use the identity

$$\int_0^\infty K_{it_f}(y)K_{it_g}(y)y^s\frac{dy}{y} = 2^{s-3}\Gamma(s)^{-1}\prod\Gamma\left(\frac{s\pm it_f\pm it_g}{2}\right)$$

to show that

$$8 \langle fg, E(\cdot, s) \rangle = \pi^{-2s} \prod \Gamma\left(\frac{s \pm it_f \pm it_g}{2}\right) L(s, f \times g)$$

where $L(s, f \times g) := \zeta(2s) \sum_{n=1}^{\infty} \frac{a(n) b(n)}{n^s}$

Deduce that $L(s, f \times g)$ is meromorphic and satisfies a functional equation. Poles can only occur at $s = 0, 1$, and only when $\langle f, g \rangle \neq 0$.

<u>Project 4</u> If f and g are arithmetically normalized even Maass forms for $SL_2(\mathbb{Z})$, then

$$L(s, f \times g) = \prod_p \prod_{i,j=1}^{2} (1 - \alpha_i(p) \beta_j(p) p^{-s})^{-1},$$

where

$$L(s, f) = \prod_p \prod_{i=1}^{2} (1 - \alpha_i(p) p^{-s})^{-1}$$

$$L(s, g) = \prod_p \prod_{j=1}^{2} (1 - \beta_j(p) p^{-s})^{-1}.$$

<u>Project 5</u> Let f, g be as in Project ~~3~~ 4. Show that $\sum_{n \leq N} a(n) b(n) = c_{fg} N + O(N^c)$ for some $c < 1$, where $c_{fg} \geq 0$ is a constant which vanishes iff $\langle f, g \rangle = 0$. Try to make c as small as possible.