Geom. Funct. Anal. Vol. 20 (2010) 1–52
DOI 10.1007/s00039-010-0063-x
Published online May 11, 2010

GAFA Geometric And Functional Analysis

TWISTED L-FUNCTIONS OVER NUMBER FIELDS AND HILBERT'S ELEVENTH PROBLEM

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Abstract. Let K be a totally real number field, π an irreducible cuspidal representation of $\mathrm{GL}_2(K)\backslash\mathrm{GL}_2(\mathbb{A}_K)$ with unitary central character, and χ a Hecke character of conductor $\mathfrak{q}$. Then $L(1/2,\pi\otimes\chi)\ll(\mathcal{N}\mathfrak{q})^{\frac{1}{2}-\frac{1}{8}(1-2\theta)+\varepsilon}$, where $0\leqslant\theta\leqslant 1/2$ is any exponent towards the Ramanujan–Petersson conjecture ($\theta=1/9$ is admissible). The proof is based on a spectral decomposition of shifted convolution sums and a generalized Kuznetsov formula.

1 Introduction

In a recent article [BlH2] the authors developed a new technique to study shifted convolution sums in Hecke eigenvalues of the type

$$\sum_{n-m=q}\lambda_{\pi_1}(n)\lambda_{\pi_2}(m)W_1(n/Y)W_2(m/Y) \tag{1}$$

for two irreducible cuspidal representations π_1,π_2 of $\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with conductor 1, reasonably regular weight functions W_1,W_2, a large number $Y>0$, and $q\neq 0$. Such sums play an important role in the theory of automorphic forms, in particular in the study of automorphic L-functions, as they constitute a typical off-diagonal term of the second moment. In this paper we generalize the method of [BlH2] to (congruence subgroups of) the Hilbert modular group of a totally real number field K, and we give applications to subconvexity of twisted L-functions over K. Before stating our main result, we note that the subconvexity problem for GL_2 over any fixed number field was recently solved by Michel and Venkatesh in a beautiful preprint [MV], where the reader can also find detailed references to previous work done in the subject. Yet, as the authors of [MV] remark, their emphasis was not on obtaining best exponents but rather finding some nontrivial exponent that works in all cases. The aim of the present paper is to demonstrate that a relatively strong

Keywords and phrases: Subconvexity, ternary quadratic forms, shifted convolution sums, spectral decomposition, Hilbert modular forms, Kirillov model

2010 *Mathematics Subject Classification*: Primary 11F41, 11F70, 11M41; Secondary 11E45, 11F72

The first author was in part supported by an NSERC grant 311664-05 and a Sloan Research Fellowship. The second author was supported by European Community grants EIF 040371 and ERG 239277 within the 6th and 7th Framework Programmes and by OTKA grants K 72731 and PD 75126.

Burgess-type subconvexity bound in the conductor aspect can be achieved for the family at hand, and once the background on automorphic forms has been set up, the proof requires comparatively little effort (cf. section 3.1).

More precisely, let π be an irreducible cuspidal representation of $\mathrm{GL}_2(K)\backslash\mathrm{GL}_2(\mathbb{A}_K)$ with unitary central character, and let χ be a Hecke character of conductor $\mathfrak{q}$. Let $L(s,\pi\otimes\chi)$ denote the twisted L-function. Let $0\leqslant\theta\leqslant 1/2$ be an approximation towards the Ramanujan–Petersson conjecture. Currently $\theta=1/9$ is known by the work of Kim and Shahidi [KiS], while the Ramanujan–Petersson conjecture predicts $\theta=0$.

Theorem 1. *For any $\varepsilon>0$ one has $L(1/2,\pi\otimes\chi)\ll_{\pi,\chi_\infty,K,\varepsilon}(\mathcal{N}\mathfrak{q})^{\frac{1}{2}-\frac{1}{8}(1-2\theta)+\varepsilon}$.*

Remark 1. This result contains a bound for all values $L(1/2+it,\,\pi\otimes\chi)$ on the critical line, because replacing $1/2$ by $1/2+it$ has the same effect as replacing χ by $\chi\otimes|\cdot|^{it}$.

Remark 2. The convexity bound in this context is $(\mathcal{N}\mathfrak{q})^{\frac{1}{2}+\varepsilon}$. The first subconvex bound over totally real number fields is a result by Cogdell, Piatetski-Shapiro and Sarnak [CoPS], [Co], in which they obtained for π induced by a holomorphic Hilbert cusp form

$$L(1/2,\pi\otimes\chi)\ll(\mathcal{N}\mathfrak{q})^{\frac{1}{2}-\frac{1-2\theta}{14+4\theta}+\varepsilon}.$$

(In this bound and in the next one we tried to optimize parameters, the original statements are somewhat weaker.) They used a very effective spectral method based on bounds for triple products [S1,2]. As an application of an ingenious and flexible geometric method, Venkatesh [V2, Th. 6.1] proved a subconvex bound over all number fields and for all irreducible cuspidal representations

$$L(1/2,\pi\otimes\chi)\ll(\mathcal{N}\mathfrak{q})^{\frac{1}{2}-\frac{(1-2\theta)^2}{14-12\theta}+\varepsilon}.$$

Our method is quite different from both of these works, and Theorem 1 supersedes both results. It may be noted that although most applications of subconvexity require only *any* nontrivial saving in the exponent, there are situations where the quality of the subconvex exponent is critical, an example being [ChCU] where $L(1/2,\pi\otimes\chi)\ll(\mathrm{cond}\,\chi)^{\frac{1}{2}-\frac{1}{16}+\varepsilon}$ or an equivalent bound for metaplectic Fourier coefficients (over $\mathbb{Q}$) is needed.

Remark 3. An inspection of the proof shows that with somewhat more precise estimates the implied constant in Theorem 1 turns out to be polynomial in the analytic conductor of π and the archimedean parameters of χ with an exponent depending on ε.

The proof of Theorem 1 builds on the ideas of several earlier works, most notably of [DFI], [CoPS], [V1,2], [BlH2]. Applying an approximate functional equation, a typical off-diagonal term in the amplified second moment is essentially of the form (1) with a slightly more general summation condition $\ell_1 n-\ell_2 m=q$ for any nonzero $q\in\mathfrak{q}$. Often the estimation of such expressions rests on some variant of the circle method (see e.g. [DFI], [BlHM]) in order to detect the summation condition. However, this seems difficult to implement over number fields with a nontrivial unit and class group, in contrast to the more structural approach in [BlH2] which we will follow here. The proof is written in an interesting mixture of classical and modern language: on the one hand, we use an adelic setup to treat the number field situation appropriately. On the other hand, at the heart of the amplification is Iwaniec's idea of playing off various subgroups against each other, and so we need to keep track carefully of the various levels occurring in the course of the argument.

Perhaps the most appealing application of Theorem 1 is to combine it with the formula of Waldspurger [W2] and its extensions by Shimura [Sh2], Khuri-Makdisi [K], Kojima [Koj], Baruch–Mao [BaM] and others in order to bound the Fourier coefficients of half-integral weight Hilbert modular forms. For $K = \mathbb{Q}$, the original breakthrough was achieved by Iwaniec [I1], and the currently strongest bounds are given in [BlH1]. For a totally real number field K other than $\mathbb{Q}$, there does not seem to be an explicit reference in the literature.

Corollary 1. *Let $(\tilde{\pi}, V_{\tilde{\pi}})$ be an irreducible cuspidal representation of $\widetilde{\mathrm{SL}}_2(K)\backslash\widetilde{\mathrm{SL}}_2(\mathbb{A}_K)$, orthogonal to one-dimensional theta series, and let $r \in \mathfrak{o}$ be a nonzero squarefree integer. Define the r-th normalized Fourier coefficient $\rho_{\tilde{\phi}}(r)$ of a pure tensor $\tilde{\phi} = \otimes_v \tilde{\phi}_v \in V_{\tilde{\pi}}$ by the left-hand side of (90) below. Then*

$$\sqrt{|\mathcal{N}r|}\rho_{\tilde{\phi}}(r) \ll_{\tilde{\phi},K,\varepsilon} |\mathcal{N}r|^{\frac{1}{4}-\frac{1}{16}(1-2\theta)+\varepsilon}.$$

Remark 4. The "trivial" bound in this context is $|\mathcal{N}r|^{\frac{1}{4}+\varepsilon}$ on the right-hand side, while the Ramanujan conjecture (implied by the Lindelöf hypothesis for twisted L-functions) states the bound $|\mathcal{N}r|^{\varepsilon}$.

One particular situation where such bounds are needed, are asymptotic formulae for the number of representations of integers by positive ternary quadratic forms, see [Bl] for an overview of this topic over $\mathbb{Q}$. Hilbert's eleventh problem asks more generally which integers are integrally represented by a given n-ary quadratic form Q over a number field K. If Q is a binary form, it corresponds to some element in the class group of a quadratic extension of K (see [Cox] for a nice account over $\mathbb{Q}$). If Q is indefinite at some archimedean place, Siegel [Si2] for $n \geqslant 4$ and Kneser [Kn] and Hsia [Hs] for $n = 3$ proved a local-to-global principle, so Siegel's mass formula [Si1] tells us exactly which integers are represented by Q. If Q is positive definite at every archimedean place and $n \geqslant 4$, again Siegel's mass formula [Si1] and bounds for Fourier coefficients of Hilbert modular forms give a complete answer (some care has to be taken in the case $n = 4$). The only remaining case of Q positive definite and $n = 3$ was solved by Duke and Schulze-Pillot [DS] for $K = \mathbb{Q}$. For arbitrary totally real K, the result was established by Cogdell, Piatetski-Shapiro and Sarnak [CoPS]; an account of the key ideas appeared in [Co]. In fact, the systematic study of subconvexity over number fields was initiated by [CoPS] about a decade ago motivated by this striking application. The relevant subconvex bound was subsequently generalized over arbitrary number fields by Venkatesh [V2], while our Corollary 1 allows a better approximation for the number of representations.

Corollary 2. *Let K be a totally real number field and let Q be a positive integral ternary quadratic form over K. Then there is an ineffective constant $c > 0$*

such that every totally positive squarefree integer $r \in \mathfrak{o}$ with $\mathcal{N}r \geqslant c$ is represented integrally by Q if and only if it is integrally represented over every completion of K. More precisely, the number of representations for such r equals $(\mathcal{N}r)^{\frac{1}{2}+o(1)} + O((\mathcal{N}r)^{\frac{7}{16}+\frac{\theta}{8}+o(1)})$, where the main term is the product of local densities given by Siegel's mass formula.

Remark 5. This result, with a slightly weaker error term, was originally proved in [CoPS]. The representation of non-squarefree integers is quite subtle, but in principle can again be characterized by more involved local considerations, cf. [Sch].

Another application of Theorem 1 can be found in [Coh, Th. 1.2] and [Z, Th. 3.2] (cf. also [V2, §1.1]) that generalizes work of Duke [D]: under the assumption of a subconvex bound as above it is proved that a certain family of Heegner points and certain d-dimensional subvarieties are equidistributed on the Hilbert modular surface $\mathrm{SL}_2(\mathfrak{o}_K)\backslash\mathcal{H}^d$.

The core of Theorem 1, from which it will follow in a fairly straightforward procedure, is the spectral decomposition of smooth shifted convolution sums which implies strong upper bounds for these sums. This is stated as Theorem 2 in section 3.2 after the necessary notation is developed. We give another application of this result in Theorem 3 of section 3.4: we prove the analytic continuation and spectral decomposition of the Dirichlet series associated to shifted convolution sums with polynomial growth on vertical lines. This problem goes back to Selberg [Se].

Section 2 contains the necessary background on automorphic forms. This section turned out to be very long; although much of the material presented there is essentially known, many of the results and computations in the number field case do not seem to be explicit in the literature. Therefore we felt that it makes the paper more useful (also a reference for future work in this subject) and readable if we give rather complete details.

Acknowledgements. This paper would not exist without the insight and guidance of Peter Sarnak who suggested Gergely Harcos in 2000 to work on this project; we are grateful for his support over the years. Important parts of this paper were written during a visit of Valentin Blomer to Rényi Institute of Mathematics in October 2008, and during the workshop "Analytic Theory of GL(3) Automorphic Forms and Applications" at the American Institute of Mathematics in November 2008. We thank both institutions for their hospitality and financial support, and the organizers of the workshop for their kind invitation. Finally we thank Jim Arthur for valuable discussions related to this work.

2 Part I: Background on Automorphic Forms

2.1 Basic Notation.

2.1.1 Number fields and adele rings. Let K be a totally real number field over $\mathbb{Q}$ of degree d, discriminant D_K, different $\mathfrak{d}$ and ring of integers $\mathfrak{o}$. Throughout the paper we regard K as fixed and all constants may depend on K, even if not stated explicitly, and they may also depend on ε which denotes an arbitrarily small positive number, not necessarily the same on each occurrence. We embed K as a $\mathbb{Q}$-algebra into $K_\infty := \mathbb{R}^d$ using the d real field embeddings $r \mapsto (r^{\sigma_1}, \dots, r^{\sigma_d})$. We denote by $K_{\infty,+}^\times := \mathbb{R}_{>0}^d$ the set of totally positive elements of K_∞, and we put

$$K_{\infty,+}^{\mathrm{diag}} := \bigl\{(x,\dots,x) \mid x \in \mathbb{R}_{>0}\bigr\} \subseteq K_{\infty,+}^\times .$$

For $r \in K$ we write

$$\mathrm{sgn}(r) := \bigl(\mathrm{sgn}(r^{\sigma_1}), \dots, \mathrm{sgn}(r^{\sigma_d})\bigr) \in \{\pm 1\}^d,$$

and we write $r >> 0$ for a totally positive integer $r \in \mathfrak{o}$. We denote by $U^+ \subseteq U$ the group of totally positive units and the group of units of $\mathfrak{o}$, respectively.

Let $\mathbb{A}$ be the adele ring of K, with K being embedded diagonally (this defines in particular a multiplication $K \times \mathbb{A} \to \mathbb{A}$). We shall often write $\mathbb{A} = K_\infty \times \mathbb{A}_{\mathrm{fin}}$. We shall label the archimedean places with elements of $\{1, \dots, d\}$ and the non-archimedean places with prime ideals $\mathfrak{p}$ of K in an obvious way. As usual, we shall denote the module of an idele $x \in \mathbb{A}^\times$ by $|x| := |x_\infty||x_{\mathrm{fin}}|$, where $|x_\infty| := \prod_{j=1}^d |x_j|$ and $|x_{\mathrm{fin}}| := \prod_{\mathfrak{p}} |x_{\mathfrak{p}}|$. We denote by $\psi : \mathbb{A} \to S^1$ the unique continuous additive character which is trivial on K, agrees with $x \mapsto e(x_1 + \dots + x_d)$ on K_∞, and on $K_{\mathfrak{p}}$ it is trivial on $\mathfrak{d}_{\mathfrak{p}}^{-1}$ but nontrivial on $\mathfrak{p}^{-1}\mathfrak{d}_{\mathfrak{p}}^{-1}$. Here and later a subscript $\mathfrak{p}$ indicates completion with respect to the corresponding valuation $v_{\mathfrak{p}}$. If $\Omega := \prod_{\mathfrak{p}} \mathfrak{o}_{\mathfrak{p}}^\times$ is the unique maximal compact subgroup of $\mathbb{A}_{\mathrm{fin}}^\times$, then $\Omega \backslash \mathbb{A}_{\mathrm{fin}}^\times$ is isomorphic in a natural way to the multiplicative group $I(K)$ of nonzero fractional ideals of K. We shall occasionally identify an idele in $\mathbb{A}_{\mathrm{fin}}^\times$ with its image under the corresponding surjective homomorphism $\mathbb{A}_{\mathrm{fin}}^\times \to I(K)$. This homomorphism also gives rise to a natural action of $\mathbb{A}_{\mathrm{fin}}^\times$ on $I(K)$. We write $\sim$ for equivalence in the ideal class group

$$C(K) := K^\times \backslash I(K) \cong K^\times \Omega \backslash \mathbb{A}_{\mathrm{fin}}^\times \cong K^\times K_\infty^\times \Omega \backslash \mathbb{A}^\times .$$

We write $h := \#C(K)$ for the class number of K. Let $\mathcal{N} : K \to \mathbb{Q}$ be the norm, which we extend to an $\mathbb{R}$-multilinear map $K_\infty \to \mathbb{R}$; the norm of a fractional ideal $\mathfrak{m} \in I(K)$ will also be denoted by $\mathcal{N}\mathfrak{m}$. Note that the norm of an infinite idele $y \in K_\infty^\times$ is $|y|$, but the norm of the fractional ideal $(y) = y\mathfrak{o} \in I(K)$ corresponding to a finite idele $y \in \mathbb{A}_{\mathrm{fin}}$ is $|y|^{-1}$.

We denote by $\mu(\mathfrak{a})$, $\varphi(\mathfrak{a})$ and $\tau(\mathfrak{a})$ the obvious generalizations of the Möbius, the Euler, and the divisor functions to nonzero ideals $\mathfrak{a} \subseteq \mathfrak{o}$. We will often use the basic estimates $\#\{\mathfrak{m} \subseteq \mathfrak{o} \mid \mathcal{N}\mathfrak{m} \leqslant x\} \asymp_K x$ for $x \geqslant 1$ and $\tau(\mathfrak{m}) \ll_{K,\varepsilon} (\mathcal{N}\mathfrak{m})^\varepsilon$ for any $\varepsilon > 0$.

2.1.2 Matrix groups. For any ring R we define the following important subgroups of $\mathrm{GL}_2(R)$:

$$Z(R) := \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \,\middle|\, a \in R^\times \right\} \quad \text{and} \quad P(R) := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \,\middle|\, a, d \in R^\times,\ b \in R \right\}.$$

For $\vartheta \in (\mathbb{R}/2\pi\mathbb{Z})^d$ we write

$$k(\vartheta) := \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} \in \mathrm{SO}_2(K_\infty) .$$

For nonzero ideals $\mathfrak{y}, \mathfrak{c} \subseteq \mathfrak{o}_\mathfrak{p}$ we define

$$\mathcal{K}(\mathfrak{y}, \mathfrak{c}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K_\mathfrak{p}) \,\middle|\, a, d \in \mathfrak{o}_\mathfrak{p},\ b \in (\mathfrak{y}\mathfrak{d}_\mathfrak{p})^{-1},\ c \in \mathfrak{y}\mathfrak{d}_\mathfrak{p}\mathfrak{c},\ ad - bc \in \mathfrak{o}_\mathfrak{p}^\times \right\}.$$

If $\mathfrak{y} = \mathfrak{o}_\mathfrak{p}$, we just write $\mathcal{K}(\mathfrak{c})$ instead of $\mathcal{K}(\mathfrak{o}_\mathfrak{p}, \mathfrak{c})$. For nonzero ideals $\mathfrak{y}, \mathfrak{c} \subseteq \mathfrak{o}$ we define

$$\mathcal{K}(\mathfrak{c}) := \prod_\mathfrak{p} \mathcal{K}(\mathfrak{c}_\mathfrak{p}) \subseteq \mathrm{GL}_2(\mathbb{A}_{\mathrm{fin}}), \quad \mathcal{K} := \mathrm{SO}_2(K_\infty) \times \mathcal{K}(\mathfrak{o}) \subseteq \mathrm{GL}_2(\mathbb{A}),$$

and

$$\Gamma(\mathfrak{y}, \mathfrak{c}) := \left\{ g_\infty \in \mathrm{GL}_2(K_\infty) \,\middle|\, g_\infty g_{\mathrm{fin}} \in \mathrm{GL}_2(K) \text{ for some } g_{\mathrm{fin}} \in \prod_\mathfrak{p} \mathcal{K}(\mathfrak{y}_\mathfrak{p}, \mathfrak{c}_\mathfrak{p}) \right\}. \quad (2)$$

The definition of $\mathcal{K}(\mathfrak{c})$ is by no means the only way of specifying a subgroup of "level $\mathfrak{c}$". We are following [Mi], [Sh1], [K] here. In [BrMP], [V1], for instance, the different $\mathfrak{d}$ is not included in the definition of $\mathcal{K}(\mathfrak{c})$, and the reader will easily see that all proofs in this paper would go through with very minor modifications, had we chosen a different definition of $\mathcal{K}(\mathfrak{c})$.

2.1.3 Measures. On K_∞ we use the normalized Lebesgue measure $|D_K|^{-1/2} dx_1 \cdots dx_d$. On $K_\mathfrak{p}$ we normalize the Haar measure so that $\mathfrak{o}_\mathfrak{p}$ has measure 1. On $\mathbb{A}$ we use the Haar measure dx which is the product of these measures, this induces the Haar probability measure on $K\backslash\mathbb{A}$. On $K_\infty^\times$ we use the Haar measure $(dy_1/|y_1|) \cdots (dy_d/|y_d|)$. On $K_\mathfrak{p}^\times$ we normalize the Haar measure so that $\mathfrak{o}_\mathfrak{p}^\times$ has measure 1. On $\mathbb{A}^\times$ we use the Haar measure $d^\times y$ which is the product of these measures, this induces some Haar measure on $K^\times\backslash\mathbb{A}^\times$. On $\mathcal{K}$ and its factors we use the Haar probability measures. On $Z(K_\infty)\backslash \mathrm{GL}_2(K_\infty)$ we use the Haar measure which satisfies

$$\int_{Z(K_\infty)\backslash \mathrm{GL}_2(K_\infty)} f(g)dg = \int_{K_\infty^\times} \int_{K_\infty} \int_{\mathrm{SO}_2(K_\infty)} f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} k \right) dk\, dx\, \frac{d^\times y}{|y|}.$$

On $\mathrm{GL}_2(K_\mathfrak{p})$ we normalize the Haar measure so that $\mathcal{K}(\mathfrak{o}_\mathfrak{p})$ has measure 1. On $Z(K_\infty)\backslash \mathrm{GL}_2(\mathbb{A})$ we use the product of these measures, this induces the Haar measure on $Z(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})$ satisfying (cf. [GJ, (3.10)])

$$\int_{Z(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})} f(g)dg = \int_{\mathbb{A}^\times} \int_{\mathbb{A}} \int_{\mathcal{K}} f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} k \right) dk\, dx\, \frac{d^\times y}{|y|}.$$

2.2 Spectral decomposition and Eisenstein series. Let $\omega : K^\times\backslash\mathbb{A}^\times \to S^1$ be a Hecke character, regarded also as a character of $Z(K)\backslash Z(\mathbb{A})$. Without loss of generality we shall assume that ω, viewed as a character of $\mathbb{A}^\times$, is trivial on $K_{\infty,+}^{\mathrm{diag}}$. (Note that in Theorem 1 replacing π by $\pi \otimes |\det|^{it}$ and χ by $\chi \otimes |\cdot|^{-it}$ leaves $\pi \otimes \chi$ unchanged. In fact for the proof of Theorem 1 we only need the results of this section for trivial ω.) The group $\mathrm{GL}_2(\mathbb{A})$ acts by right translation on the Hilbert space

$$L^2\big(\mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbb{A}), \omega\big)$$

of measurable functions $\phi : \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$ satisfying

$$\phi(\gamma z g) = \omega(z)\phi(g), \quad \gamma \in \mathrm{GL}_2(K),\ z \in Z(\mathbb{A}),\ g \in \mathrm{GL}_2(\mathbb{A}),$$

$$\langle \phi, \phi \rangle := \int_{\mathrm{GL}_2(K)Z(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})} |\phi(g)|^2 dg < \infty.$$

A function $\phi \in L^2(\mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbb{A}),\omega)$ is called cuspidal if

$$\int_{K\backslash\mathbb{A}} \phi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right) dx = 0 \quad \text{for almost all } g \in \mathrm{GL}_2(\mathbb{A})\,.$$

We have a $\mathrm{GL}_2(\mathbb{A})$-invariant decomposition

$$L^2\big(\mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbb{A}),\omega\big) = L_{\mathrm{cusp}} \oplus L_{\mathrm{cusp}}^{\perp}$$

into the space of cuspidal functions and its orthogonal complement. The cuspidal space decomposes into a Hilbert space direct sum of irreducible automorphic representations (here and later we do not indicate closure for notational simplicity):

$$L_{\mathrm{cusp}} = \bigoplus_{\pi\in\mathcal{C}_\omega} V_\pi\,.$$

The orthogonal complement $L_{\mathrm{cusp}}^{\perp}$ is described in detail in [GJ, §3-5], see also [Bu, §3.7]: For any Hecke character χ satisfying $\chi^2=\omega$ (which is necessarily trivial on $K_{\infty,+}^{\mathrm{diag}}$) let V_χ be the subspace generated by the function $g\mapsto\chi(\det g)$, then we have a $\mathrm{GL}_2(\mathbb{A})$-invariant (orthogonal) decomposition

$$L_{\mathrm{cusp}}^{\perp} = L_{\mathrm{sp}} \oplus L_{\mathrm{cont}}\,, \quad L_{\mathrm{sp}} := \bigoplus_{\chi^2=\omega} V_\chi\,,$$

where L_{cont} can be described as follows.

For two Hecke quasicharacters $\chi_1,\chi_2 : K^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ with $\chi_1\chi_2=\omega$ let $H(\chi_1,\chi_2)$ denote the space of functions $\varphi:\mathrm{GL}_2(\mathbb{A})\to\mathbb{C}$ satisfying

$$\int_{\mathcal{K}} |\varphi(k)|^2 dk < \infty$$

and

$$\varphi\left(\begin{pmatrix}a & x\\ 0 & b\end{pmatrix} g\right) = \chi_1(a)\chi_2(b)\left|\frac{a}{b}\right|^{1/2}\varphi(g)\,, \quad x\in\mathbb{A}\,,\ a,b\in\mathbb{A}^\times. \tag{3}$$

We can regard this as the space of functions $\varphi\in L^2(\mathcal{K})$ satisfying

$$\varphi\left(\begin{pmatrix}a & x\\ 0 & b\end{pmatrix} k\right) = \chi_1(a)\chi_2(b)\varphi(k)\,, \quad \begin{pmatrix}a & x\\ 0 & b\end{pmatrix}\in\mathcal{K}\,. \tag{4}$$

There is a unique $s\in\mathbb{C}$ such that $\chi_1(a)=|a|^s$ and $\chi_2(a)=|a|^{-s}$ for all $a\in K_{\infty,+}^{\mathrm{diag}}$. Accordingly, for $s\in\mathbb{C}$ we introduce

$$H(s) := \bigoplus_{\substack{\chi_1\chi_2=\omega\\ \chi_1\chi_2^{-1}=|\cdot|^{2s}\text{ on }K_{\infty,+}^{\mathrm{diag}}}} H(\chi_1,\chi_2)\,, \tag{5}$$

and we view the space $H:=\int_{\mathbb{C}} H(s)ds$ as a holomorphic fibre bundle with base $\mathbb{C}$. (Note that on [GJ, p. 224], $\mu\circ\nu^{-1}(a)=|a|^s$ should read $\mu\circ\nu^{-1}(a)=|a|^{2s}$, cf. [GJ, (3.11)].) For a section $\varphi\in H$ we use the obvious notation $\varphi(s)\in H(s)$ and $\varphi(s,g)\in\mathbb{C}$. The bundle H is trivial, because any $\varphi(s_0)\in H(s_0)$ extends uniquely to a section $\varphi\in H$. There is a $\mathrm{GL}_2(\mathbb{A})$-equivariant isomorphism

$$S: L_{\mathrm{cont}} \to L'_{\mathrm{cont}} := \int_0^\infty H(iy)dy\,,$$

given explicitly by [GJ, (4.23)] on a dense subspace. If we equip L'_{cont} with the inner product

$$\langle\varphi_1,\varphi_2\rangle := \frac{2}{\pi}\int_0^\infty \big\langle\varphi_1(iy),\varphi_2(iy)\big\rangle dy = \frac{2}{\pi}\int_0^\infty\int_{\mathcal{K}}\varphi_1(iy,k)\bar{\varphi}_2(iy,k)dk\,dy\,,$$

then this map is an isometry by [GJ, §4, Part D]; in combination with the theory of Eisenstein series [GJ, §5] it yields a spectral decomposition of L_{cont}. For a section $\varphi\in H$ and for $g\in\mathrm{GL}_2(\mathbb{A})$ we define the Eisenstein series

$$E\big(\varphi(s),g\big) := \sum_{\gamma\in P(K)\backslash\mathrm{GL}_2(K)}\varphi(s,\gamma g)\,,\quad \Re s>1/2\,. \tag{6}$$

This is a holomorphic function which extends meromorphically to $s\in\mathbb{C}$ with no poles on $\Re s=0$. Moreover, for any $s\neq 0$ which is not a pole of $E(\varphi(s),g)$, we can extract $\varphi(s)\in H(s)$ from the meromorphic continuation of the constant term as given by [GJ, (5.3)]. The above discussion suggests that for $y\in\mathbb{R}^\times$ we consider the complex vector space

$$V(iy) := \big\{E(\varphi(iy))\mid\varphi(iy)\in H(iy)\big\}$$

equipped with the inner product

$$\big\langle E(\varphi_1(iy)),E(\varphi_2(iy))\big\rangle := \big\langle\varphi_1(iy),\varphi_2(iy)\big\rangle\,. \tag{7}$$

By (5) we have a $\mathrm{GL}_2(\mathbb{A})$-invariant (orthogonal) decomposition

$$V(iy) = \bigoplus_{\substack{\chi_1\chi_2=\omega\\ \chi_1\chi_2^{-1}=|\cdot|^{2iy}\text{ on }K_{\infty,+}^{\text{diag}}}} V_{\chi_1,\chi_2}\,,$$

where

$$V_{\chi_1,\chi_2} := \big\{E(\varphi(iy))\mid\varphi(iy)\in H(\chi_1,\chi_2)\big\}\,.$$

We note that $V_{\chi_1,\chi_2}=V_{\chi_2,\chi_1}$, in particular $V(iy)=V(-iy)$, by [GJ, (4.3), (4.24), (5.15)]. Now we have a $\mathrm{GL}_2(\mathbb{A})$-invariant decomposition

$$L_{\text{cont}} = \int_0^\infty V(iy)dy = \int_0^\infty \bigoplus_{\substack{\chi_1\chi_2=\omega\\ \chi_1\chi_2^{-1}=|\cdot|^{2iy}\text{ on }K_{\infty,+}^{\text{diag}}}} V_{\chi_1,\chi_2}dy\,.$$

More precisely, by [GJ, (4.24), (5.15)–(5.17)] any $\phi\in L_{\text{cont}}$ can be written as

$$\phi(g) = \frac{1}{\pi}\int_0^\infty E\big(\varphi(iy),g\big)dy\,,\quad \varphi:=S\phi\in L'_{\text{cont}}\,,$$

and we have Plancherel's identity

$$\begin{aligned}\langle\phi_1,\phi_2\rangle &= \frac{1}{\pi}\int_0^\infty\big\langle E(\varphi_1(iy)),\phi_2\big\rangle dy = \frac{1}{\pi}\int_0^\infty 2\big\langle\varphi_1(iy),\varphi_2(iy)\big\rangle dy\\ &= \frac{2}{\pi}\int_0^\infty\big\langle E(\varphi_1(iy)),E(\varphi_2(iy))\big\rangle dy\,.\end{aligned}$$

To summarize, we have a $\mathrm{GL}_2(\mathbb{A})$-invariant orthogonal decomposition

$$L^2\big(\mathrm{GL}_2(K)\backslash\mathrm{GL}_2(\mathbb{A}),\omega\big) = \bigoplus_{\pi\in\mathcal{C}_\omega}V_\pi\oplus\bigoplus_{\chi^2=\omega}V_\chi\oplus\int_0^\infty\bigoplus_{\substack{\chi_1\chi_2=\omega\\ \chi_1\chi_2^{-1}=|\cdot|^{2iy}\text{ on }K_{\infty,+}^{\text{diag}}}}V_{\chi_1,\chi_2}dy \tag{8}$$

in the sense that each function in the L^2-space decomposes into a convergent sum and integral of functions from each subspace, and a Plancherel formula holds. For notational simplicity we shall write the last term as $\int_{\mathcal{E}_\omega} V_\varpi \, d\varpi$, where $\mathcal{E}_\omega$ is the set of *unordered pairs* $\{\chi_1,\chi_2\}$ of Hecke characters with $\chi_1\chi_2=\omega$ and nontrivial restrictions on $K_{\infty,+}^{\mathrm{diag}}$.

2.3 Casimir eigenvalues and conductors. Let (π, V_π) be an infinite-dimensional irreducible automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ occurring in the spectral decomposition (8), i.e. one of V_π with $\pi\in\mathcal{C}_\omega$, or V_{χ_1,χ_2} with $\{\chi_1,\chi_2\}\in\mathcal{E}_\omega$, equipped with the right $\mathrm{GL}_2(\mathbb{A})$-action. By Flath's theorem [F], V_π decomposes as a restricted tensor product over the places of K,

$$V_\pi=\bigotimes_v V_{\pi_v}. \tag{9}$$

For each $1\leqslant j\leqslant d$, the Laplace–Beltrami operator of the j-th component of $\mathrm{GL}_2(K_\infty)=\mathrm{GL}_2(\mathbb{R})^d$,

$$\Delta_j:=-y_j^2(\partial_{x_j}^2+\partial_{y_j}^2)+y_j\partial_{x_j}\partial_{\vartheta_j}, \tag{10}$$

acts on the dense subset V_π^∞ of smooth vectors by a scalar

$$\lambda_{\pi,j}=:\tfrac14-\nu_{\pi,j}^2\in\mathbb{R}.$$

Here $\nu_{\pi,j}\in\frac12\mathbb{Z}$ if $\pi_{\infty,j}$ belongs to the discrete series, and by [KiS] we have

$$\nu_{\pi,j}\in i\mathbb{R}\cup[-\theta,\theta], \quad \theta:=1/9, \tag{11}$$

if $\pi_{\infty,j}$ belongs to the principal series or the complementary series. We shall choose $\nu_{\pi,j}$ so that $\Re\nu_{\pi,j}\geqslant 0$ and $\Im\nu_{\pi,j}\geqslant 0$. For notational simplicity we write

$$\begin{aligned}\lambda_\pi&:=(\lambda_{\pi,j})_{j=1}^d\in\mathbb{R}^d, &\quad \tilde\lambda_\pi&:=\bigl(1+|\lambda_{\pi,j}|\bigr)_{j=1}^d\in\mathbb{R}_{>0}^d;\\ \nu_\pi&:=(\nu_{\pi,j})_{j=1}^d\in\mathbb{R}^d, &\quad \tilde\nu_\pi&:=\bigl(1+|\nu_{\pi,j}|\bigr)_{j=1}^d\in\mathbb{R}_{>0}^d;\end{aligned} \tag{12}$$

in particular,

$$\mathcal{N}\tilde\lambda_\pi=\prod_{j=1}^d\bigl(1+|\lambda_{\pi,j}|\bigr) \quad\text{and}\quad \mathcal{N}\tilde\nu_\pi=\prod_{j=1}^d\bigl(1+|\nu_{\pi,j}|\bigr).$$

Let $\mathfrak{c}_\omega\subseteq\mathfrak{o}$ denote the conductor of the central character ω, and for a nonzero ideal $\mathfrak{c}\subseteq\mathfrak{c}_\omega$ let

$$V_\pi(\mathfrak{c}):=\left\{\phi\in V_\pi \;\middle|\; \phi\left(g\begin{pmatrix}a&b\\c&d\end{pmatrix}\right)=\omega_{\mathfrak{c}}(d)\phi(g) \text{ for all } g\in\mathrm{GL}_2(\mathbb{A}) \text{ and } \begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathcal{K}(\mathfrak{c})\right\},$$

where

$$\omega_{\mathfrak{c}}(x):=\prod_{\mathfrak{p}\mid\mathfrak{c}}\omega_{\mathfrak{p}}(x), \quad x\in\mathbb{A}^\times.$$

For $\mathfrak{c}\subseteq\mathfrak{c}'\subseteq\mathfrak{c}_\omega$ we have $V_\pi(\mathfrak{c}')\subseteq V_\pi(\mathfrak{c})$, because $\omega_{\mathfrak{c}}(d)=\omega_{\mathfrak{c}'}(d)$ for $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathcal{K}(\mathfrak{c})$. (Indeed, for $\mathfrak{p}\mid\mathfrak{c}$ and $\mathfrak{p}\nmid\mathfrak{c}'$ we have $bc\in\mathfrak{p}\mathfrak{o}_{\mathfrak{p}}$, hence $a,d\in\mathfrak{o}_{\mathfrak{p}}^\times$ by $ad-bc\in\mathfrak{o}_{\mathfrak{p}}^\times$, so that $\omega_{\mathfrak{p}}(d)=1$ by $\mathfrak{p}\nmid\mathfrak{c}_\omega$.) We define the conductor $\mathfrak{c}_\pi$ of π as the largest ideal $\mathfrak{c}\subseteq\mathfrak{c}_\omega$

such that $V_\pi(\mathfrak{c}) \neq \{0\}$ (cf. [C, Th. 1] and [Mi, Cor. 2]). Analogously, for a prime $\mathfrak{p}$ and a nonzero ideal $\mathfrak{c} \subseteq \mathfrak{c}_{\omega_\mathfrak{p}}$ we define $V_{\pi_\mathfrak{p}}(\mathfrak{c})$ and the local conductor $\mathfrak{c}_{\pi_\mathfrak{p}}$. Note that $\mathfrak{c}_{\pi_\mathfrak{p}} = \mathfrak{c}_\pi \mathfrak{o}_\mathfrak{p}$. Finally, we define the analytic conductor of π (cf. [IS]) as

$$C(\pi) := (\mathcal{N}\mathfrak{c}_\pi)(\mathcal{N}\tilde{\lambda}_\pi)\,. \tag{13}$$

For any nonzero ideals $\mathfrak{t}, \mathfrak{c} \subseteq \mathfrak{o}$ such that $\mathfrak{t}\mathfrak{c}_\pi \mid \mathfrak{c}$ there is an isometric embedding of complex vector spaces

$$R_\mathfrak{t} : V_\pi(\mathfrak{c}_\pi) \hookrightarrow V_\pi(\mathfrak{c})\,, \quad (R_\mathfrak{t}\phi)(g) := \phi\left(g\begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right), \tag{14}$$

where $t \in \mathbb{A}_{\mathrm{fin}}^\times$ is any finite idele representing $\mathfrak{t}$. It follows from (9) and the local result of Casselman [C] that the spaces $V_\pi(\mathfrak{c})$ decompose (in general not orthogonally) as

$$V_\pi(\mathfrak{c}) = \bigoplus_{\mathfrak{t} \mid \mathfrak{c}\mathfrak{c}_\pi^{-1}} R_\mathfrak{t} V_\pi(\mathfrak{c}_\pi)\,, \quad \text{for any } \mathfrak{c} \subseteq \mathfrak{c}_\pi\,, \tag{15}$$

and $V_{\pi_\mathfrak{p}}(c_{\pi_\mathfrak{p}})$ is one-dimensional for each prime $\mathfrak{p}$. For each character $k(\vartheta) \mapsto \exp(iq\cdot\vartheta)$, $q \in \mathbb{Z}^d$, of $\mathrm{SO}_2(K_\infty)$ we define

$$V_{\pi,q} := \left\{\phi \in V_\pi \mid \phi(gk(\vartheta)) = \exp(iq\cdot\vartheta)\phi(g) \text{ for all } \vartheta \in (\mathbb{R}/2\pi\mathbb{Z})^d\right\},$$

and correspondingly we write

$$V_{\pi,q}(\mathfrak{c}) := V_{\pi,q} \cap V_\pi(\mathfrak{c})\,.$$

This gives an orthogonal decomposition (in a Hilbert space sense)

$$V_\pi(\mathfrak{c}) = \bigoplus_{q \in \mathbb{Z}^d} V_{\pi,q}(\mathfrak{c})\,, \quad \text{for any } \mathfrak{c} \subseteq \mathfrak{c}_\pi\,, \tag{16}$$

and also a decomposition of vector spaces

$$V_{\pi,q}(\mathfrak{c}) = \bigoplus_{\mathfrak{t} \mid \mathfrak{c}\mathfrak{c}_\pi^{-1}} R_\mathfrak{t} V_{\pi,q}(\mathfrak{c}_\pi)\,, \quad \text{for any } \mathfrak{c} \subseteq \mathfrak{c}_\pi\,, \tag{17}$$

where $V_{\pi,q}(\mathfrak{c}_\pi)$ is at most one-dimensional. Alternately, (17) and global multiplicity-one were also established by Miyake [Mi] which then imply the above local results.

In the case of $V_\pi = V_{\chi_1,\chi_2}$ consisting of Eisenstein series we replace the subscript π by χ_1, χ_2 for convenience, e.g. we write $\mathfrak{c}_{\chi_1,\chi_2} := \mathfrak{c}_\pi$. For each $1 \leqslant j \leqslant d$ we have

$$\lambda_{\chi_1,\chi_2,j} = \tfrac{1}{4} - s_j^2\,, \quad \nu_{\chi_1,\chi_2,j} = \pm s_j\,, \tag{18}$$

where $s_j \in i\mathbb{R}$ denotes the unique exponent such that $\chi_1\chi_2^{-1} = |\cdot|^{2s_j}$ on the j-th component of $K_{\infty,+}^\times$. We note that $\chi_1\chi_2^{-1} = |\cdot|^{2iy}$ on $K_{\infty,+}^{\mathrm{diag}}$, where

$$y := \frac{s_1 + s_2 + \cdots + s_d}{id} \in \mathbb{R}^\times. \tag{19}$$

It follows from the discussion in section 2.2 that V_{χ_1,χ_2} and $H(\chi_1,\chi_2)$ are isomorphic representations, in particular there is a decomposition

$$H(\chi_1,\chi_2) = \bigotimes_v H_v(\chi_1,\chi_2)\,. \tag{20}$$

In addition,

$$V_{\chi_1,\chi_2}(\mathfrak{c}) = \left\{E(\varphi(iy),\,\cdot\,) \in V_{\chi_1,\chi_2} \mid \varphi \in H(\chi_1,\chi_2,\mathfrak{c})\right\}, \tag{21}$$

and by (15),

$$H(\chi_1,\chi_2,\mathfrak{c}) = \bigoplus_{\mathfrak{t}\mid\mathfrak{c}\mathfrak{c}_\pi^{-1}} R_{\mathfrak{t}} H(\chi_1,\chi_2,\mathfrak{c}_\pi)\,.$$

Here it is known (e.g. [C, p. 306]) that $\mathfrak{c}_\pi = \mathfrak{c}_{\chi_1,\chi_2} = \mathfrak{c}_{\chi_1}\mathfrak{c}_{\chi_2}$. In section 2.6 we shall give a detailed proof of this fact for trivial central character.

Finally for $\mathfrak{c} \subseteq \mathfrak{c}_\omega$ we define, in harmony with the notation of the previous section,

$$\begin{aligned} \mathcal{C}_\omega(\mathfrak{c}) &:= \big\{\pi \in \mathcal{C}_\omega \mid \mathfrak{c} \subseteq \mathfrak{c}_\pi \subseteq \mathfrak{c}_\omega\big\}\,,\\ \mathcal{E}_\omega(\mathfrak{c}) &:= \big\{\{\chi_1,\chi_2\} \in \mathcal{E}_\omega \mid \mathfrak{c} \subseteq \mathfrak{c}_{\chi_1,\chi_2} \subseteq \mathfrak{c}_\omega\big\}\,, \end{aligned} \tag{22}$$

and we shall drop the subscript ω in case ω is trivial.

2.4 Normalized Whittaker functions. Let $q \in \mathbb{Z}$. For q even, let $\nu \in \big(\frac12+\mathbb{Z}\big) \cup i\mathbb{R} \cup \big(-\frac12,\frac12\big)$. For q odd, let $\nu \in \mathbb{Z} \cup i\mathbb{R}$. For these parameters we define the normalized Whittaker function

$$\tilde W_{q/2,\nu}(y) := \frac{i^{\operatorname{sgn}(y)\frac q2} W_{\operatorname{sgn}(y)\frac q2,\nu}(4\pi|y|)}{\big\{\Gamma\big(\frac12-\nu+\operatorname{sgn}(y)\frac q2\big)\Gamma\big(\frac12+\nu+\operatorname{sgn}(y)\frac q2\big)\big\}^{1/2}}\,, \quad y \in \mathbb{R}^\times, \tag{23}$$

where $W_{\alpha,\beta}$ is the standard Whittaker function, see [WhW, Ch. XVI]. The right-hand side is understood as 0 if one of $\frac12 \pm \nu + \operatorname{sgn}(y)\frac q2$ is a nonpositive integer, otherwise we have a positive number under the square-root sign by the constraints on ν. We note that the above definition is invariant under $\nu \to -\nu$, and for future reference we record that

$$\tilde W_{\frac q2,\nu}(y) = \eta_{q,\nu}\frac{W_{\operatorname{sgn}(y)\frac q2,\nu}(4\pi|y|)}{\Gamma\big(\frac12+\nu+\operatorname{sgn}(y)\frac q2\big)}\,, \quad q \in 2\mathbb{Z}\,,\ \nu \in i\mathbb{R}\,,\ y \in \mathbb{R}^\times, \tag{24}$$

where $\eta_{q,\nu}$ is a constant of modulus 1 depending on q and ν but not on y. (This can be proved by induction on q, starting from the trivial case $q=0$. Note that (24) is only stated for special q and ν.)

By [BrMo, §4], the functions $\tilde W_{q/2,\nu}$ ($q \in \mathbb{Z}$) for fixed ν form an orthonormal basis of the Hilbert space $L^2(\mathbb{R}^\times, d^\times y)$ which justifies our normalization:

$$L^2(\mathbb{R}^\times, d^\times y) = \bigoplus_{q\in\mathbb{Z}} \mathbb{C}\tilde W_{\frac q2,\nu}\,, \quad \langle \tilde W_{\frac q2,\nu}, \tilde W_{\frac{q'}{2},\nu}\rangle = \delta_{q,q'}\,. \tag{25}$$

We review the uniform bounds [BlH2, (24)–(26)]. For all ν we have

$$\tilde W_{q/2,\nu}(y) \ll |y|^{1/2}\left(\frac{|y|}{|q|+|\nu|+1}\right)^{-1-|\Re\nu|}\exp\left(-\frac{|y|}{|q|+|\nu|+1}\right). \tag{26}$$

For $\nu \in \frac12\mathbb{Z} \cup i\mathbb{R}$ we have, for any $0<\varepsilon<1/4$,

$$\tilde W_{q/2,\nu}(y) \ll_\varepsilon |y|^{1/2-\varepsilon}\big(|q|+|\nu|+1\big)\,. \tag{27}$$

For $\nu \in (-1/2,1/2)$ we have, for any $0<\varepsilon<1$,

$$\tilde W_{q/2,\nu}(y) \ll_\varepsilon |y|^{1/2-|\nu|-\varepsilon}\big(|q|+|\nu|+1\big)^{1+|\nu|}. \tag{28}$$

For $q \in \mathbb{Z}^d$ and appropriate $\nu \in \mathbb{C}^d$, we define

$$\tilde W_{q/2,\nu}(y) := \prod_{j=1}^d \tilde W_{q_j/2,\nu_j}(y_j)\,, \quad y \in K_\infty^\times\,. \tag{29}$$

2.5 Hecke eigenvalues and Fourier expansion. Let $\mathfrak{c} \subseteq \mathfrak{c}_\pi$ be a nonzero ideal. There is a character $\varepsilon_\pi : \{\pm 1\}^d \to \{\pm 1\}$ depending only on π such that every $\phi \in V_{\pi,q}(\mathfrak{c})$ has a Fourier–Whittaker expansion (cf. [K, (2.11), (3.8)])

$$\phi\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \rho_{\phi,0}(y) + \sum_{r\in K^\times} \rho_\phi(ry_{\rm fin})\varepsilon_\pi\bigl(\operatorname{sgn}(ry_\infty)\bigr)\tilde{W}_{q/2,\nu_\pi}(ry_\infty)\psi(rx) \qquad (30)$$

for $y = y_\infty \times y_{\rm fin} \in \mathbb{A}^\times$, $x \in \mathbb{A}$. Note that for any $y_{\rm fin} \in \mathbb{A}_{\rm fin}^\times$ the coefficient $\rho_\phi(y_{\rm fin})$ only depends on the fractional ideal represented by $y_{\rm fin}$ and it is nonzero only if this ideal is integral. The normalization of $\tilde{W}_{q/2,\nu_\pi}$ is further justified by the fact that these coefficients remain unchanged if ϕ is replaced by any of its nonzero Maaß shifts.

If (π, V_π) is one of the right $\mathrm{GL}_2(\mathbb{A})$-spaces V_χ with $\chi \in X$ in (8), then the expansion (30) only contains the constant term $\rho_0(y)$.

Let us now assume that (π, V_π) is one of the right $\mathrm{GL}_2(\mathbb{A})$-spaces V_π with $\pi \in \mathcal{C}_\omega$ in (8), so that $\rho_0(y) = 0$. The finer structure of the coefficients ρ_ϕ can be revealed by the theory of Hecke operators, as developed by Miyake [Mi] (see also [Sh1, §2] and [K, §2]). By (17) we can decompose any vector $\phi \in V_{\pi,q}(\mathfrak{c})$ as

$$\phi = \sum_{\mathfrak{t}\mid\mathfrak{c}\mathfrak{c}_\pi^{-1}} R_\mathfrak{t}\phi_\mathfrak{t}\,,$$

where each $\phi_\mathfrak{t}$ lies in $V_{\pi,q}(\mathfrak{c}_\pi)$. By (30) we infer

$$\rho_\phi(\mathfrak{m}) = \sum_{\mathfrak{t}\mid\mathfrak{c}\mathfrak{c}_\pi^{-1}} \rho_{R_\mathfrak{t}\phi_\mathfrak{t}}(\mathfrak{m}) = \sum_{\mathfrak{t}\mid\mathfrak{c}\mathfrak{c}_\pi^{-1}} \rho_{\phi_\mathfrak{t}}(\mathfrak{m}\mathfrak{t}^{-1})\,, \quad \mathfrak{m} \subseteq \mathfrak{o}\,,$$

so that we can focus our attention to the case when $\mathfrak{c} = \mathfrak{c}_\pi$, i.e. when $\phi \in V_{\pi,q}(\mathfrak{c}_\pi)$ is a newform. For each nonzero ideal $\mathfrak{m} \subseteq \mathfrak{o}$ the Hecke operator $T_{\mathfrak{c}_\pi}(\mathfrak{m})$ acts on $V_\pi(\mathfrak{c}_\pi)$ by a scalar $\lambda_\pi(\mathfrak{m})$. The function λ_π satisfies

$$\lambda_\pi(\mathfrak{m})\lambda_\pi(\mathfrak{n}) = \sum_{\mathfrak{a}\mid\gcd(\mathfrak{m},\mathfrak{n})} \omega_\pi(\mathfrak{a})\lambda_\pi(\mathfrak{m}\mathfrak{n}\mathfrak{a}^{-2})\,, \qquad (31)$$

and

$$\lambda_\pi(\mathfrak{m}) = \omega_\pi(\mathfrak{m})\bar{\lambda}_\pi(\mathfrak{m})\,, \quad \gcd(\mathfrak{m},\mathfrak{c}_\pi) = \mathfrak{o}\,, \qquad (32)$$

where $\omega_\pi : I(K) \to \mathbb{C}$ is defined as follows: if $\mathfrak{a} \in I(K)$ is coprime to $\mathfrak{c}_\pi$ then $\omega_\pi(\mathfrak{a}) := \omega(a)$ where $a \in \mathbb{A}_{\rm fin}^\times$ is any finite idele representing $\mathfrak{a}$ with $a_\mathfrak{p} = 1$ for $\mathfrak{p} \mid \mathfrak{c}_\pi$, otherwise $\omega_\pi(\mathfrak{a}) := 0$. In particular, λ_π is multiplicative on the set of nonzero integral ideals. The non-archimedean analogue of (11) is

$$\lambda_\pi(\mathfrak{m}) \ll_\varepsilon (N\mathfrak{m})^{\theta+\varepsilon}, \quad \theta := 1/9\,, \qquad (33)$$

for any $\varepsilon > 0$, see [KiS]. It follows, as stated after (17), that each $V_{\pi,q}(\mathfrak{c}_\pi)$ is at most one-dimensional, and in fact

$$\rho_\phi(\mathfrak{m}) = \frac{\lambda_\pi(\mathfrak{m})}{\sqrt{\mathcal{N}\mathfrak{m}}}\rho_\phi(\mathfrak{o})\,, \quad \mathfrak{m} \subseteq \mathfrak{o},\ \phi \in V_{\pi,q}(\mathfrak{c}_\pi)\,.$$

To maintain this identity we define $\lambda_\pi(\mathfrak{a})$ to be zero for any nonintegral $\mathfrak{a} \in I(K)$. Comparing with (30) we see that for $\phi \in V_{\pi,q}(\mathfrak{c}_\pi)$ we have

$$\phi\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \sum_{r\in K^\times} \frac{\lambda_\pi(ry_{\rm fin})}{\sqrt{\mathcal{N}(ry_{\rm fin})}} W_\phi(ry_\infty)\psi(rx)\,, \quad y \in \mathbb{A}^\times,\ x \in \mathbb{A}\,, \qquad (34)$$

where

$$W_\phi(y) = \rho_\phi(\mathfrak{o})\varepsilon_\pi(\mathrm{sgn}(y))\tilde{W}_{q/2,\nu_\pi}(y)\,, \quad y \in K_\infty^\times\,,\ \phi \in V_{\pi,q}(\mathfrak{c}_\pi)\,. \tag{35}$$

An intrinsic definition of W_ϕ becomes apparent upon choosing $y_{\mathrm{fin}} = (1, 1, \dots)$ and $x_{\mathrm{fin}} = (0, 0, \dots)$ in (34) and picking by orthogonality and $\mathrm{vol}(K\backslash\mathbb{A}) = 1$ the term corresponding to $r = 1$:

$$W_\phi(y) := \int_{K\backslash\mathbb{A}} \phi\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right)\psi(-x)dx\,, \quad y \in K_\infty^\times. \tag{36}$$

We have verified (34) and (36) for pure weight newforms $\phi \in V_{\pi,q}(\mathfrak{c}_\pi)$ but then, by linearity, it extends to all smooth vectors $\phi \in V_\pi^\infty(\mathfrak{c}_\pi)$. Using also (25), we obtain a linear map from $V_\pi^\infty(\mathfrak{c}_\pi)$ to a dense subspace of $L^2(K_\infty^\times, d^\times y)$ given by $\phi \mapsto W_\phi$. We will prove in section 2.9 that

$$\langle\phi_1, \phi_2\rangle = C_\pi\langle W_{\phi_1}, W_{\phi_2}\rangle \tag{37}$$

for some positive constant C_π depending only on π. It follows that the map $\phi \mapsto W_\phi$ extends to a vector space isomorphism $V_\pi(\mathfrak{c}_\pi) \to L^2(K_\infty^\times, d^\times y)$, called the (archimedean) Kirillov map of π, and Lemma 3 below shows that it is essentially an isometry (i.e. $C_\pi \approx 1$). In particular, (34) and (37) hold for all $\phi \in V_\pi(\mathfrak{c}_\pi)$.

Now let $\mathfrak{c} \subseteq \mathfrak{c}_\pi$ be any ideal. It will be important to investigate in detail vectors in the larger space $V_\pi(\mathfrak{c})$, classically called "oldforms". The proofs of the following facts depend partly on the theory of Eisenstein series that we will develop in later sections (independently of the present statements, of course).

As mentioned earlier, the decomposition (15) is in general not orthogonal. However, by a Gram–Schmidt orthogonalization process based on (80) below, we find for each pair of integral ideals $(\mathfrak{s}, \mathfrak{t})$ with $\mathfrak{s} \mid \mathfrak{t} \mid \mathfrak{c}\mathfrak{c}_\pi^{-1}$ complex numbers $\alpha_{\mathfrak{t},\mathfrak{s}}$ such that

$$R^{(\mathfrak{t})} := \sum_{\mathfrak{s}\mid\mathfrak{t}} \alpha_{\mathfrak{t},\mathfrak{s}} R_{\mathfrak{s}} : V_\pi(\mathfrak{c}_\pi) \hookrightarrow V_\pi(\mathfrak{c})\,, \quad \mathfrak{t} \mid \mathfrak{c}\mathfrak{c}_\pi^{-1}, \tag{38}$$

are isometric embeddings with pairwise orthogonal images, and $R^{(\mathfrak{o})}$ is the identical inclusion map. This yields an orthogonal decomposition

$$V_\pi(\mathfrak{c}) = \bigoplus_{\mathfrak{t}\mid\mathfrak{c}\mathfrak{c}_\pi^{-1}} R^{(\mathfrak{t})}V_\pi(\mathfrak{c}_\pi)\,, \quad \text{for any } \mathfrak{c} \subseteq \mathfrak{c}_\pi\,, \tag{39}$$

and an extension of the Kirillov map (36) to each subspace $R^{(\mathfrak{t})}V_\pi(\mathfrak{c}_\pi)$. Namely, by (34) every $\phi \in R^{(\mathfrak{t})}V_\pi(\mathfrak{c}_\pi)$ has a Fourier expansion

$$\phi\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \sum_{r\in K^\times} \frac{\lambda_\pi^{(\mathfrak{t})}(ry_{\mathrm{fin}})}{\sqrt{\mathcal{N}(ry_{\mathrm{fin}})}} W_\phi(ry_\infty)\psi(rx)\,, \quad y \in \mathbb{A}^\times,\ x \in \mathbb{A}\,, \tag{40}$$

where

$$W_\phi := W_{(R^{(\mathfrak{t})})^{-1}\phi} \quad \text{and} \quad \lambda_\pi^{(\mathfrak{t})}(\mathfrak{m}) := \sum_{\mathfrak{s}\mid\gcd(\mathfrak{t},\mathfrak{m})} \alpha_{\mathfrak{t},\mathfrak{s}}(\mathcal{N}\mathfrak{s})^{1/2}\lambda_\pi(\mathfrak{m}\mathfrak{s}^{-1})\,. \tag{41}$$

It is clear that (37) holds true when extended to $\phi_1, \phi_2 \in R^{(\mathfrak{t})}V_\pi(\mathfrak{c}_\pi)$, and Lemma 3 below shows that

$$C(\pi)^{-\varepsilon}\|\phi\| \ll_{K,\varepsilon} \|W_\phi\| \ll_{K,\varepsilon} C(\pi)^\varepsilon\|\phi\|\,, \quad \phi \in R^{(\mathfrak{t})}V_\pi(\mathfrak{c}_\pi)\,, \tag{42}$$

with implied constants depending only on K and ε.

REMARK 6. If $\mathfrak{c}$ is squarefree, then the orthogonalization can be carried out completely explicitly by combining (80) below with the Hecke relations (31)–(32) above (see e.g. [ILS, Prop. 2.6]), and one obtains $\alpha_{\mathfrak{t},\mathfrak{s}} \ll_\varepsilon (\mathcal{N}\mathfrak{t}\mathfrak{s}^{-1})^{\theta-1/2+\varepsilon}$. For general ideals $\mathfrak{c}$, this seems much harder.

2.6 Parametrizing Eisenstein series. For simplicity we shall assume in the following three sections that the central character ω is trivial, since this is all we need for our purposes. The general case, however, is quite similar. We can assume that $\chi_1 = \chi$ and $\chi_2 = \chi^{-1}$, where χ is a Hecke character which is nontrivial on $K_{\infty,+}^{\mathrm{diag}}$. Let us denote the conductor of χ by $\mathfrak{c}_\chi$, and for an arbitrary place v of K let us write χ_v for the restriction of χ to the quasifactor $K_v^\times$ of $\mathbb{A}^\times$. Note that $\mathfrak{c}_{\chi_\mathfrak{p}} = \mathfrak{c}_\chi\mathfrak{o}_\mathfrak{p}$ for each prime $\mathfrak{p}$. For every prime $\mathfrak{p}$ we fix a prime element $\varpi_\mathfrak{p} \in \mathfrak{o}_\mathfrak{p}$ (i.e. $v_\mathfrak{p}(\varpi_\mathfrak{p}) = 1$) and we shall use the convention that $v_\mathfrak{p}(0) = \infty$. For the purpose of this paper we could get by with less information than provided in this section, but we have preferred to give rather precise results.

LEMMA 1. *The conductor of $H(\chi,\chi^{-1})$ is $\mathfrak{c}_\chi^2$. More precisely, let $\mathfrak{p}$ be a prime, $\varpi := \varpi_\mathfrak{p}$, $r := v_\mathfrak{p}(\mathfrak{d})$, $m := v_\mathfrak{p}(\mathfrak{c}_\chi)$. For any integer $n \geqslant 0$ the complex vector space $H_\mathfrak{p}(\chi,\chi^{-1},\mathfrak{p}^n)$ has dimension* $\max(0, n-2m+1)$. *For $n \geqslant 2m$ an orthogonal basis is $\{\varphi_{\mathfrak{p},j} \mid 0 \leqslant j \leqslant n-2m\}$, where the functions $\varphi_{\mathfrak{p},j} : \mathcal{K}(\mathfrak{o}_\mathfrak{p}) \to \mathbb{C}$ are defined as follows.*

- *When $m = 0$ (i.e. χ is unramified at $\mathfrak{p}$) and $k = \begin{pmatrix} * & * \\ b\varpi^r & * \end{pmatrix} \in \mathcal{K}(\mathfrak{o}_\mathfrak{p})$,*

$$\varphi_{\mathfrak{p},0}(k) := 1\,; \quad \varphi_{\mathfrak{p},1}(k) := \begin{cases} (\mathcal{N}\mathfrak{p})^{-1/2}, & v_\mathfrak{p}(b) = 0\,, \\ -(\mathcal{N}\mathfrak{p})^{1/2}, & v_\mathfrak{p}(b) \geqslant 1\,; \end{cases} \tag{43}$$

  and for $j \geqslant 2$,

$$\varphi_{\mathfrak{p},j}(k) := \begin{cases} 0\,, & v_\mathfrak{p}(b) \leqslant j-2\,, \\ -(\mathcal{N}\mathfrak{p})^{j/2-1}, & v_\mathfrak{p}(b) = j-1\,, \\ (\mathcal{N}\mathfrak{p})^{j/2}\left(1 - \frac{1}{\mathcal{N}\mathfrak{p}}\right), & v_\mathfrak{p}(b) \geqslant j\,. \end{cases} \tag{44}$$

- *When $m > 0$ (i.e. χ is ramified at $\mathfrak{p}$) and $k = \begin{pmatrix} a & * \\ b\varpi^r & * \end{pmatrix} \in \mathcal{K}(\mathfrak{o}_\mathfrak{p})$,*

$$\varphi_{\mathfrak{p},j}(k) := \begin{cases} (\mathcal{N}\mathfrak{p})^{(m+j)/2}\chi_\mathfrak{p}(ab^{-1})\,, & v_\mathfrak{p}(b) = m+j\,, \\ 0\,, & v_\mathfrak{p}(b) \neq m+j\,. \end{cases} \tag{45}$$

REMARK 7. The basis exhibited above is close to orthonormal. Using

$$\left[\mathcal{K}(\mathfrak{o}_\mathfrak{p}) : \mathcal{K}((\varpi^j))\right] = (\mathcal{N}\mathfrak{p})^j\left(1 + \frac{1}{\mathcal{N}\mathfrak{p}}\right), \quad j \geqslant 1\,,$$

it is straightforward to see that

$$1 - \frac{1}{\mathcal{N}\mathfrak{p}} \leqslant \|\varphi_{p,j}\| \leqslant 1\,, \quad j \geqslant 0\,, \tag{46}$$

with equality on the right-hand side for $j = 0$.

Proof. For the argument below it is useful to keep in mind that for any nonzero ideal $\mathfrak{c} \subseteq \mathfrak{o}_\mathfrak{p}$

$$\mathcal{K}(\mathfrak{c}) = \begin{pmatrix} \varpi^r & 0 \\ 0 & 1 \end{pmatrix}^{-1} \mathcal{K}_0(\mathfrak{c}) \begin{pmatrix} \varpi^r & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{K}_0(\mathfrak{c}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathfrak{o}_\mathfrak{p}) \;\Big|\; c \in \mathfrak{c} \right\}.$$

In particular, $\mathcal{K}(\mathfrak{c})$ has the same measure as $\mathcal{K}_0(\mathfrak{c})$.

We can regard $H_\mathfrak{p}(\chi, \chi^{-1}, \mathfrak{p}^n)$ as a subset of functions on $N(\mathfrak{o}_\mathfrak{p})\backslash\mathcal{K}(\mathfrak{o}_\mathfrak{p})/\mathcal{K}((\varpi^n))$ with $N(\mathfrak{o}_\mathfrak{p}) := \{ (\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}) \mid x \in \mathfrak{d}_\mathfrak{p}^{-1} \}$. A set of double coset representatives for $N(\mathfrak{o}_\mathfrak{p})\backslash\mathcal{K}(\mathfrak{o}_\mathfrak{p})/\mathcal{K}((\varpi^n))$ is given by any collection

$$\left\{ \begin{pmatrix} a & * \\ \varpi^{r+j} & * \end{pmatrix} \in \mathcal{K}(\mathfrak{o}_\mathfrak{p}) \;\Big|\; 0 \leqslant j \leqslant n\,,\ a \in \mathfrak{o}_\mathfrak{p}^\times,\ a \bmod \varpi^{\min(j,n-j)} \right\},$$

where for given a and j any choice of $*$ is admissible. To see this, we observe first that by [Sh3, Proof on p. 25 and Errata on p. 269], this set has the right cardinality. Moreover, two such representatives determine different double cosets. Indeed, multiplying a representative from the left by elements of $N(\mathfrak{o}_\mathfrak{p})$ and from the right by elements of $\mathcal{K}((\varpi^n))$ does not change the valuation of the lower left entry if $j < n$, and it can at most increase the valuation if $j = n$, but all representatives have $j \leqslant n$, so we conclude that different values of j correspond to different double cosets. In addition, if $(\begin{smallmatrix} a & * \\ \varpi^{r+j} & * \end{smallmatrix})$ and $(\begin{smallmatrix} a' & * \\ \varpi^{r+j} & * \end{smallmatrix})$ are in the same double coset, then

$$\begin{pmatrix} a' & * \\ \varpi^{r+j} & * \end{pmatrix}^{-1} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & * \\ \varpi^{r+j} & * \end{pmatrix} \in \mathcal{K}\big((\varpi^n)\big)\,,$$

whence $a'\varpi^j - a\varpi^j - x\varpi^{2j} \in (\varpi^n)$. This forces $a = a'$ if $0 < j < n$, whereas in the remaining cases $j = 0$ and $j = n$ only $a = a' = 1$ is allowed.

By a variant of the argument above we see that each $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \mathcal{K}(\mathfrak{o}_\mathfrak{p})$ is represented by some $(\begin{smallmatrix} a' & * \\ \varpi^{r+j} & * \end{smallmatrix}) \in \mathcal{K}(\mathfrak{o}_\mathfrak{p})$ with $a' \in \mathfrak{o}_\mathfrak{p}^\times$ and $j = \min(v_\mathfrak{p}(c) - r, n)$. Now for any $\varphi \in H_\mathfrak{p}(\chi, \chi^{-1}, \mathfrak{p}^n)$ the transformation rule (4) shows that

$$\varphi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} a' & * \\ \varpi^{r+j} & * \end{pmatrix}\right) = \chi_\mathfrak{p}(a')\varphi\left(\begin{pmatrix} 1 & * \\ \varpi^{r+j} & * \end{pmatrix}\right), \quad a \in \mathfrak{o}_\mathfrak{p}^\times,$$

hence φ is determined by the $n+1$ values $\varphi\big((\begin{smallmatrix} 1 & * \\ \varpi^{r+j} & * \end{smallmatrix})\big)$ with $0 \leqslant j \leqslant n$. By the discussion of representatives we can further see that $\varphi\big((\begin{smallmatrix} 1 & * \\ \varpi^{r+j} & * \end{smallmatrix})\big) \neq 0$ implies $\chi_\mathfrak{p}(a') = 1$ for any $a' \in 1 + (\varpi^{\min(j,n-j)})$, i.e. $m \leqslant j \leqslant n - m$. In other words, φ is determined by the $n - 2m + 1$ values $\varphi\big((\begin{smallmatrix} 1 & * \\ \varpi^{r+j} & * \end{smallmatrix})\big)$ with $m \leqslant j \leqslant n - m$, because the rest of the $n+1$ values are zero. The dependence on these values is linear, hence the dimension of $H_\mathfrak{p}(\chi, \chi^{-1}, \mathfrak{p}^n)$ is at most $n - 2m + 1$. For $m = 0$, it is straightforward to check that the $n+1$ functions $\varphi_{\mathfrak{p},j}$ for $0 \leqslant j \leqslant n$ defined by (43) and (44) lie in $H_\mathfrak{p}(\chi, \chi^{-1}, \mathfrak{p}^n)$ and, by $[\mathcal{K}(\mathfrak{o}_\mathfrak{p}) : \mathcal{K}((\varpi^j))] = (\mathcal{N}\mathfrak{p})^j\big(1 + \frac{1}{\mathcal{N}\mathfrak{p}}\big)$ for $j \geqslant 1$ they are pairwise orthogonal. For $m > 0$, it is straightforward to check that the $n - 2m + 1$ functions $\varphi_{\mathfrak{p},j}$ for $0 \leqslant j \leqslant n - 2m$ defined by (45) lie in $H_\mathfrak{p}(\chi, \chi^{-1}, \mathfrak{p}^n)$ and they are pairwise orthogonal because their supports are pairwise disjoint. Any orthogonal system is linearly independent, hence the proof of the lemma is complete. □

The lemma can be combined with (20) to obtain an orthogonal basis of $H(\chi, \chi^{-1}, \mathfrak{c})$ for each ideal $\mathfrak{c} \subseteq \mathfrak{c}_\chi^2$. Namely, for any $\mathfrak{t} \mid \mathfrak{c}\mathfrak{c}_\chi^{-2}$ and any $q \in (2\mathbb{Z})^d$, we define

$\varphi^{(\mathfrak{t},q)} : \mathcal{K} \to \mathbb{C}$ to be the tensor product of the local functions $\varphi_\infty^{(\mathfrak{t},q)}(k(\vartheta)) := e^{iq\vartheta}$ and $\varphi_\mathfrak{p}^{(\mathfrak{t},q)} := \varphi_{\mathfrak{p},v_\mathfrak{p}(\mathfrak{t})}$ as in the lemma. These global functions form an orthogonal basis of $H(\chi,\chi^{-1},\mathfrak{c})$; extending them to $\mathrm{GL}_2(\mathbb{A})$ by (3), the corresponding vectors $\phi^{(\mathfrak{t},q)} := E(\varphi^{(\mathfrak{t},q)})$ form an orthogonal basis of $V_{\chi,\chi^{-1}}(\mathfrak{c})$ by (21) and (7). We obtain isometric embeddings $R^{(\mathfrak{t})} : V_{\chi,\chi^{-1}}(\mathfrak{c}_\chi^2) \hookrightarrow V_{\chi,\chi^{-1}}(\mathfrak{c})$ by defining $R^{(\mathfrak{t})} : \phi^{(\mathfrak{o},q)}/\|\phi^{(\mathfrak{o},q)}\| \mapsto \phi^{(\mathfrak{t},q)}/\|\phi^{(\mathfrak{t},q)}\|$ for all $q \in (2\mathbb{Z})^d$. These yield an orthogonal decomposition

$$V_{\chi,\chi^{-1}}(\mathfrak{c}) = \bigoplus_{\mathfrak{t}\mid\mathfrak{c}\mathfrak{c}_\chi^{-2}} R^{(\mathfrak{t})} V_{\chi,\chi^{-1}}(\mathfrak{c}_\chi^2)\,, \quad \text{for any } \mathfrak{c} \subseteq \mathfrak{c}_\chi^2\,, \tag{47}$$

similarly as in the cuspidal case, see (39). In the next section we shall exhibit for each nonzero ideal $\mathfrak{t} \subseteq \mathfrak{o}$ a vector space isomorphism $R^{(\mathfrak{t})}V_{\chi,\chi^{-1}}(\mathfrak{c}_\chi^2) \to L^2(K_\infty^\times, d^\times y)$, written as $\phi \mapsto W_\phi$, such that every $\phi \in R^{(\mathfrak{t})}V_{\chi,\chi^{-1}}(\mathfrak{c}_\chi^2)$ has a Fourier expansion with similar features as in the cuspidal case (cf. (40)–(42)):

$$\phi\left(\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}\right) = \rho_{\phi,0}(y) + \sum_{r\in K^\times} \frac{\lambda_{\chi,\chi^{-1}}^{(\mathfrak{t})}(ry_{\mathrm{fin}})}{\sqrt{\mathcal{N}(ry_{\mathrm{fin}})}} W_\phi(ry_\infty)\psi(rx)\,, \quad y \in \mathbb{A}^\times,\ x \in \mathbb{A}\,, \tag{48}$$

where

$$\lambda_{\chi,\chi^{-1}}^{(\mathfrak{t})}(\mathfrak{m}) \ll_{K,\varepsilon} \big(\mathcal{N}\gcd(\mathfrak{t},\mathfrak{m})\big)(\mathcal{N}\mathfrak{m})^\varepsilon, \quad \mathfrak{m} \subseteq \mathfrak{o}\,, \tag{49}$$

$$\|W_\phi\| \ll_{K,\varepsilon} (\mathcal{N}\mathfrak{t})^\varepsilon C(\chi,\chi^{-1})^\varepsilon \|\phi\|\,, \quad \phi \in R^{(\mathfrak{t})}V_{\chi,\chi^{-1}}(\mathfrak{c}_\chi^2)\,. \tag{50}$$

Next we prove a density result about the Eisenstein spectrum.

LEMMA 2. *For a nonzero ideal $\mathfrak{c} \subseteq \mathfrak{o}$ write $\mathfrak{c} = \mathfrak{c}_1^2\mathfrak{c}_2$ with $\mathfrak{c}_2$ squarefree. In the notation of* (18) *and* (22) *we have, for any $X \geqslant 1$,*

$$\int_{\substack{\varpi\in\mathcal{E}_\omega(\mathfrak{c})\\ |\nu_{\varpi,j}|\leqslant X}} 1\, d\varpi \ll_K X^d(\mathcal{N}\mathfrak{c}_1)\,.$$

Proof. We need to estimate the measure of the set of Hecke character pairs $\{\chi,\chi^{-1}\}$ for which $\mathfrak{c} \subseteq \mathfrak{c}_{\chi,\chi^{-1}}$ and $\chi = |\cdot|^{s_j}$ with some $s_j \in i[-X,X]$ on the j-th component of $K_{\infty,+}^\times$. We write $\chi = \chi_\infty\chi_{\mathrm{fin}}$ with the obvious convention. By Lemma 1, $\mathfrak{c}_\chi$ must divide $\mathfrak{c}_1$, hence the number of possibilities for the restriction of χ_{fin} to $\Omega = \prod_\mathfrak{p} \mathfrak{o}_\mathfrak{p}^\times$ is at most $\varphi(\mathfrak{c}_1)$. We fix therefore any character $\xi : \Omega \to S^1$ and estimate the measure of the set of Hecke character pairs $\{\chi,\chi^{-1}\}$ for which χ_{fin} agrees with ξ on Ω and $(s_1,\dots,s_d) \in i[-X,X]^d$. If this set is empty, we are done, otherwise we fix some element χ_0 in it. Any χ in the set has the feature that $\tilde\chi := \chi\chi_0^{-1}$ is trivial on Ω, moreover $\tilde\chi = |\cdot|^{s_j}$ on the j-th component of $K_{\infty,+}^\times$ with $(s_1,\dots,s_d) \in i\mathcal{B}$, where $\mathcal{B} := [-2X,2X]^d$. As $\tilde\chi$ is also trivial on $K^\times$, its infinite part $\tilde\chi_\infty$ is trivial on the set of totally positive units U^+ embedded in $K_{\infty,+}^\times$. Let $\{u_1,\dots,u_{d-1}\}$ be a generating set of U^+, and put

$$M := \begin{pmatrix} 1 & \cdots & 1 \\ \log u_1^{\sigma_1} & \cdots & \log u_1^{\sigma_d} \\ \vdots & & \vdots \\ \log u_{d-1}^{\sigma_1} & \cdots & \log u_{d-1}^{\sigma_d}\end{pmatrix} \in \mathbb{R}^{d\times d}.$$

Then $s \in i\mathcal{B}$, regarded as a column vector, satisfies $Ms \in i\Lambda(y)$, where $y \in [-2X, 2X]$ is as in (19) and $\Lambda(y) := \{yd\} \times (2\pi\mathbb{Z})^{d-1}$ is a lattice in an affine hyperplane of $\mathbb{R}^d$. Note that M is non-singular and depends only the field K, hence by the finiteness of $K^\times K_{\infty,+}^\times \Omega \backslash \mathbb{A}^\times$ it suffices to show that

$$\int_{-2X}^{2X} \#\big(\Lambda(y) \cap M\mathcal{B}\big) dy \ll_K X^d.$$

The integrand is $\ll_K X^{d-1}$, hence the required bound follows. □

2.7 Explicit Fourier expansion of Eisenstein series. The aim of this section is to verify the relations (48)–(50). In particular, we need to define $W_\phi : K_\infty \to \mathbb{C}$ for each $\phi \in R^{(\mathfrak{t})} V_{\chi,\chi^{-1}}(\mathfrak{c}_\chi^2)$ and identify the coefficients $\lambda_{\chi,\chi^{-1}}^{(\mathfrak{t})}(\mathfrak{m})$ for nonzero ideals $\mathfrak{m} \subseteq \mathfrak{o}$. By linearity and orthogonality, we can assume that ϕ is one of the pure tensors $\phi^{(\mathfrak{t},q)} := E(\varphi^{(\mathfrak{t},q)}) \in R^{(\mathfrak{t})} V_{\chi,\chi^{-1}}(\mathfrak{c}_\chi^2)$ introduced after the proof of Lemma 1. The superscript indicates a nonzero ideal $\mathfrak{t} \subseteq \mathfrak{o}$ and a vector $q \in (2\mathbb{Z})^d$: we shall keep it fixed and drop it from the notation for simplicity.

First of all we observe that (46) implies

$$(\mathcal{N}\mathfrak{t})^{-\varepsilon} \ll_{K,\varepsilon} \|\varphi\| \leqslant 1 \tag{51}$$

for any $\varepsilon > 0$. We insert the definition (6) into the Fourier decomposition (strictly speaking, we should extend φ to a section $\varphi(s) \in H(s)$, perform the calculation for $\Re s > 1/2$, and deduce the result for $\Re s \geqslant 0$ by meromorphic continuation; we also note that $\mathrm{vol}(K\backslash\mathbb{A}) = 1$ by our choice of Haar measures)

$$E\left(\varphi, \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \sum_{r \in K} \int_{K\backslash\mathbb{A}} E\left(\varphi, \begin{pmatrix} y & \xi \\ 0 & 1 \end{pmatrix}\right) \psi(-r\xi) d\xi \, \psi(rx)\,,$$

and use the fact that by the Bruhat decomposition a complete set of representatives of $P(K)\backslash \mathrm{GL}_2(K)$ is given by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and the matrices $\left(\begin{smallmatrix} 0 & -1 \\ 1 & * \end{smallmatrix}\right)$ in $\mathrm{GL}_2(K)$. We obtain

$$E\left(\varphi, \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) + \sum_{r \in K} \int_{\mathbb{A}} \varphi\left(\begin{pmatrix} 0 & -1 \\ y & \xi \end{pmatrix}\right) \psi(-r\xi) d\xi \, \psi(rx)\,.$$

On the right-hand side $r = 0$ contributes to the constant term of $E(\varphi)$. From now on we shall assume that $r \in K^\times$. We define $\delta \in \mathbb{A}^\times$ by $\delta_\infty := (1, \ldots, 1)$ and $\delta_\mathfrak{p} := \varpi_\mathfrak{p}^{v_\mathfrak{p}(\mathfrak{d})}$ for each prime $\mathfrak{p}$. By the change of variable $\xi \to y\delta^{-1}\xi$ the integral over $\mathbb{A}$ becomes, using also (3) and $\chi(r) = |r| = 1$,

$$\chi^2(\delta)\chi^{-1}(ry)|ry|^{1/2} \int_{\mathbb{A}} \varphi\left(\begin{pmatrix} 0 & -\delta^{-1} \\ \delta & \xi \end{pmatrix}\right) \psi(-ry\delta^{-1}\xi) d\xi\,. \tag{52}$$

As φ is a pure tensor, we can write this expression as a product of local factors in a natural fashion.

By our choice of the Haar measure on K_∞, the infinite part of (52) is $|D_K|^{-1/2}$ times the product over $1 \leqslant j \leqslant d$ of the following expressions:

$$\chi_j(\mathrm{sgn}(ry))|r^{\sigma_j} y_j|^{1/2-s_j} \int_{-\infty}^{\infty} \varphi_j\left(\begin{pmatrix} 0 & -1 \\ 1 & \xi \end{pmatrix}\right) e(-r^{\sigma_j} y_j \xi) d\xi\,, \tag{53}$$

where $s_j \in i\mathbb{R}$ denotes the unique exponent such that $\chi_j = |\cdot|^{s_j}$ on $\mathbb{R}_{>0}$ (cf. notation following (18)). Using the Iwasawa decomposition

$$\begin{pmatrix} 0 & -1 \\ 1 & \xi \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\xi^2+1}} & \frac{-\xi}{\sqrt{\xi^2+1}} \\ 0 & \sqrt{\xi^2+1} \end{pmatrix} \begin{pmatrix} \frac{\xi}{\sqrt{\xi^2+1}} & \frac{-1}{\sqrt{\xi^2+1}} \\ \frac{1}{\sqrt{\xi^2+1}} & \frac{\xi}{\sqrt{\xi^2+1}} \end{pmatrix}, \tag{54}$$

we see that

$$\varphi_j\left(\begin{pmatrix} 0 & -1 \\ 1 & \xi \end{pmatrix}\right) = \frac{\chi_j^{-1}(\xi^2+1)}{\sqrt{\xi^2+1}} \left(\frac{\xi - i}{\sqrt{\xi^2+1}}\right)^{q_j} = \frac{1}{(\xi^2+1)^{1/2+s_j}} \left(\frac{\xi-i}{\xi+i}\right)^{q_j/2},$$

where we used that q_j is even. Therefore (53) equals

$$\chi_j(\operatorname{sgn}(ry))|r^{\sigma_j} y_j|^{1/2-s_j} \int_{-\infty}^{\infty} \frac{e(-r^{\sigma_j} y_j \xi)}{(\xi^2+1)^{1/2+s_j}} \left(\frac{\xi-i}{\xi+i}\right)^{q_j/2} d\xi\,.$$

The integral on the right-hand side remains unchanged when $r^{\sigma_j} y_j$ and q_j are replaced by $|r^{\sigma_j} y_j|$ and $\operatorname{sgn}(r^{\sigma_j} y_j) q_j$, hence by [BrMo, (2.16)] we deduce (after the change of variable $\xi \to -\xi$) that the previous display equals

$$\chi_j(\operatorname{sgn}(ry))(-1)^{\frac{q_j}{2}} \pi^{\frac{1}{2}+s_j} \frac{W_{\operatorname{sgn}(r^{\sigma_j} y_j)\frac{q_j}{2}, s_j}(4\pi |r^{\sigma_j} y_j|)}{\Gamma\big(\frac{1}{2} + s_j + \operatorname{sgn}(r^{\sigma_j} y_j)\frac{q_j}{2}\big)}\,.$$

Now we can combine (18), (24), (29) to conclude that the infinite part of (52) can be written as

$$\eta_\infty \chi_\infty\big(\operatorname{sgn}(r y_\infty)\big) \tilde{W}_{q/2, \nu_{\chi,\chi^{-1}}}(r y_\infty)\,, \tag{55}$$

where $\eta_\infty = \eta_\infty(q, \chi_\infty) \in \mathbb{C}$ is a constant of modulus $\pi^{d/2}|D_K|^{-1/2}$.

We now calculate the local factor of (52) corresponding to a prime $\mathfrak{p}$. For simplicity we shall omit the subscripts in $\varpi_\mathfrak{p}$ and $\delta_\mathfrak{p}$. The calculation is based on the following Iwasawa decomposition for $\xi \neq 0$:

$$\begin{pmatrix} 0 & -\delta^{-1} \\ \delta & \xi \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\delta^{-1} \\ \delta & \xi \end{pmatrix}, & v_\mathfrak{p}(\xi) \geqslant 0\,, \\ \begin{pmatrix} \xi^{-1} & -\delta^{-1} \\ 0 & \xi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \delta\xi^{-1} & 1 \end{pmatrix}, & v_\mathfrak{p}(\xi) < 0\,. \end{cases} \tag{56}$$

We write $m := v_\mathfrak{p}(\mathfrak{c}_\chi) \geqslant 0$, $n := v_\mathfrak{p}(ry) \geqslant 0$, and we recall that $\varphi_\mathfrak{p} = \varphi_{\mathfrak{p}, v_\mathfrak{p}(\mathfrak{t})}$ is as in Lemma 1.

We first consider the case when $m = 0$, i.e. $\chi_\mathfrak{p}$ is unramified. We assume that $v_\mathfrak{p}(\mathfrak{t}) = 0$, then $\varphi_\mathfrak{p}$ is constant 1 on $\mathcal{K}(\mathfrak{o}_\mathfrak{p})$, and by (56) we have

$$\begin{aligned} \int_{K_\mathfrak{p}} \varphi_\mathfrak{p}&\left(\begin{pmatrix} 0 - \delta^{-1} & \\ \delta & \xi \end{pmatrix}\right) \psi(-ry\delta^{-1}\xi) d\xi \\ &= \int_{v_\mathfrak{p}(\xi) \geqslant 0} \psi(-ry\delta^{-1}\xi) d\xi + \int_{v_\mathfrak{p}(\xi) < 0} \chi_\mathfrak{p}^{-2}(\xi)|\xi|^{-1} \psi(-ry\delta^{-1}\xi) d\xi \\ &= 1 + \sum_{j=1}^{\infty} \chi_\mathfrak{p}(\varpi^{2j})(\mathcal{N}\mathfrak{p})^{-j} \int_{v_\mathfrak{p}(\xi) = -j} \psi(-ry\delta^{-1}\xi) d\xi\,. \end{aligned}$$

We calculate the inner integral by observing

$$\int_{v_\mathfrak{p}(\xi)=-j}\psi(-ry\delta^{-1}\xi)d\xi=\begin{cases}(\mathcal{N}\mathfrak{p})^j\left(1-\frac{1}{\mathcal{N}\mathfrak{p}}\right), & 1\leqslant j\leqslant n\,,\\ -(\mathcal{N}\mathfrak{p})^n, & j=n+1\,,\\ 0\,, & j\geqslant n+2\,.\end{cases}\tag{57}$$

We obtain

$$\begin{aligned}\int_{K_\mathfrak{p}}\varphi_\mathfrak{p}\left(\begin{pmatrix}0 & -\delta^{-1}\\ \delta & \xi\end{pmatrix}\right)\psi(-ry\delta^{-1}\xi)d\xi &= 1-\frac{\chi_\mathfrak{p}(\varpi^{2n+2})}{\mathcal{N}\mathfrak{p}}+\left(1-\frac{1}{\mathcal{N}\mathfrak{p}}\right)\sum_{j=1}^{n}\chi_\mathfrak{p}(\varpi^{2j})\\ &=\left(1-\frac{\chi_\mathfrak{p}(\varpi^2)}{\mathcal{N}\mathfrak{p}}\right)\sum_{j=0}^{n}\chi_\mathfrak{p}(\varpi^{2j})\,.\end{aligned}$$

We proved that for $\chi_\mathfrak{p}$ unramified and $v_\mathfrak{p}(\mathfrak{t})=0$ the local factor of (52) corresponding to $\mathfrak{p}$ equals

$$|ry|_\mathfrak{p}^{1/2}\chi_\mathfrak{p}^2(\delta)\left(1-\frac{\chi_\mathfrak{p}^2(\varpi)}{\mathcal{N}\mathfrak{p}}\right)\sum_{j=0}^{n}\chi_\mathfrak{p}(\varpi^{2j-n})\,.\tag{58}$$

For $v_\mathfrak{p}(\mathfrak{t})=1$ a similar calculation based on (43) and (56) shows that

$$\begin{aligned}&\int_{K_\mathfrak{p}}\varphi_\mathfrak{p}\left(\begin{pmatrix}0 & -\delta^{-1}\\ \delta & \xi\end{pmatrix}\right)\psi(-ry\delta^{-1}\xi)d\xi\\ &\qquad=\frac{1+\chi_\mathfrak{p}(\varpi^{2n+2})}{(\mathcal{N}\mathfrak{p})^{1/2}}-(\mathcal{N}\mathfrak{p})^{1/2}\left(1-\frac{1}{\mathcal{N}\mathfrak{p}}\right)\sum_{j=1}^{n}\chi_\mathfrak{p}(\varpi^{2j})\,,\end{aligned}$$

where we understand any empty sum as zero. If $\chi_\mathfrak{p}^2(\varpi)\neq-1$, then we conclude that the local factor of (52) corresponding to $\mathfrak{p}$ has absolute value equal to $|1+\chi_\mathfrak{p}^2(\varpi)|(\mathcal{N}\mathfrak{p})^{-1/2}$ for $n=0$ and not exceeding $(n+1)(\mathcal{N}\mathfrak{p})^{(1-n)/2}$ in general. If $\chi_\mathfrak{p}^2(\varpi)=-1$, then we conclude that the local factor of (52) corresponding to $\mathfrak{p}$ equals

$$|ry|_\mathfrak{p}^{1/2}\chi_\mathfrak{p}^2(\delta)\chi_\mathfrak{p}^{-1}(\varpi)(\mathcal{N}\mathfrak{p})^{1/2}\left(1-\frac{\chi_\mathfrak{p}^2(\varpi)}{\mathcal{N}\mathfrak{p}}\right)\sum_{j=0}^{n-1}\chi_\mathfrak{p}(\varpi^{2j-n+1})\,,\tag{59}$$

an expression very similar to (58). For $v_\mathfrak{p}(\mathfrak{t})\geqslant 2$ a similar calculation based on (44) and (56) shows that

$$\begin{aligned}&\int_{K_\mathfrak{p}}\varphi_\mathfrak{p}\left(\begin{pmatrix}0 & -\delta^{-1}\\ \delta & \xi\end{pmatrix}\right)\psi(-ry\delta^{-1}\xi)d\xi\\ &\quad=-\chi_\mathfrak{p}\big(\varpi^{2v_\mathfrak{p}(\mathfrak{t})-2}\big)(\mathcal{N}\mathfrak{p})^{-v_\mathfrak{p}(\mathfrak{t})/2}\int_{v_\mathfrak{p}(\xi)=1-v_\mathfrak{p}(\mathfrak{t})}\psi(-ry\delta^{-1}\xi)d\xi\\ &\qquad+\left(1-\frac{1}{\mathcal{N}\mathfrak{p}}\right)\sum_{j=v_\mathfrak{p}(\mathfrak{t})}^{\infty}\chi_\mathfrak{p}(\varpi^{2j})(\mathcal{N}\mathfrak{p})^{v_\mathfrak{p}(\mathfrak{t})/2-j}\int_{v_\mathfrak{p}(\xi)=-j}\psi(-ry\delta^{-1}\xi)d\xi\,.\end{aligned}$$

Using (57), we conclude that the local factor of (52) corresponding to $\mathfrak{p}$ vanishes for $n\leqslant v_\mathfrak{p}(\mathfrak{t})-3$, has absolute value equal to $(\mathcal{N}\mathfrak{p})^{-1}$ for $n=v_\mathfrak{p}(\mathfrak{t})-2$ and not exceeding $(n-v_\mathfrak{p}(\mathfrak{t})+3)(\mathcal{N}\mathfrak{p})^{(v_\mathfrak{p}(\mathfrak{t})-n)/2}$ for $n\geqslant v_\mathfrak{p}(\mathfrak{t})-1$.

We turn to the case when $m > 0$, i.e. $\chi_\mathfrak{p}$ is ramified. We combine (45) with (56) to see that the local factor of (52) corresponding to $\mathfrak{p}$ equals

$$\chi_\mathfrak{p}^2(\delta)\chi_\mathfrak{p}^{-1}(ry)|ry|_\mathfrak{p}^{1/2}(\mathcal{N}\mathfrak{p})^{(m+v_\mathfrak{p}(\mathfrak{t}))/2}\int_{v_\mathfrak{p}(\xi)=-m-v_\mathfrak{p}(\mathfrak{t})}\chi_\mathfrak{p}^{-2}(\xi)|\xi|^{-1}\chi_\mathfrak{p}(\xi)\,\psi(-ry\delta^{-1}\xi)d\xi\,.$$

By the change of variable $\xi \to (ry)^{-1}\xi$ this is the same as

$$\chi_\mathfrak{p}^2(\delta)(\mathcal{N}\mathfrak{p})^{(n-m-v_\mathfrak{p}(\mathfrak{t}))/2}\int_{v_\mathfrak{p}(\xi)=n-m-v_\mathfrak{p}(\mathfrak{t})}\chi_\mathfrak{p}^{-1}(\xi)\psi(-\delta^{-1}\xi)d\xi\,.$$

For $n = v_\mathfrak{p}(\mathfrak{t})$ the integral is a Gauß sum of absolute value $(\mathcal{N}\mathfrak{p})^{m/2}$. For $n < v_\mathfrak{p}(\mathfrak{t})$ we pick $z \in \mathfrak{p}^{-1}\mathfrak{o}_\mathfrak{p}^\times$ such that $\psi(\delta^{-1}z) \neq 1$. Changing ξ to $\xi + z = \xi(1+\xi^{-1}z)$ does not affect the value $\chi_\mathfrak{p}^{-1}(\xi)$ and therefore introduces an additional factor $\psi(-\delta^{-1}z) \neq 1$, hence the integral vanishes. For $n > v_\mathfrak{p}(\mathfrak{t})$ we pick $z \in 1 + \mathfrak{p}^{m-1}\mathfrak{o}_\mathfrak{p}^\times$ if $m > 1$ and $z \in \mathfrak{o}_\mathfrak{p}^\times$ if $m = 1$ such that $\chi_\mathfrak{p}(z) \neq 1$. Changing ξ to $\xi z = \xi - \xi(1-z)$ does not affect the value $\psi(-\delta^{-1}\xi)$ and therefore introduces an additional factor $\chi_\mathfrak{p}^{-1}(z) \neq 1$, hence again the integral vanishes. We proved that for $\chi_\mathfrak{p}$ ramified the local factor of (52) corresponding to $\mathfrak{p}$ equals

$$\begin{cases}\eta_\mathfrak{p}\,, & v_\mathfrak{p}(ry) = v_\mathfrak{p}(\mathfrak{t})\,,\\ 0\,, & v_\mathfrak{p}(ry) \neq v_\mathfrak{p}(\mathfrak{t})\,,\end{cases} \tag{60}$$

where $\eta_\mathfrak{p} = \eta_\mathfrak{p}(\chi_\mathfrak{p}, \varpi_\mathfrak{p}) \in \mathbb{C}$ is a constant of modulus 1.

By the above discussion (in particular by (52), (55), (58), (59), (60)), we can see that the Fourier coefficients $\rho_{E(\varphi)}(\mathfrak{m})$ in (30) are supported on ideals $\mathfrak{m}$ divisible by

$$\mathfrak{t}_\chi := \prod_{\substack{\mathfrak{p}|\mathfrak{t},\,\mathfrak{p}\nmid\mathfrak{c}_\chi\\ v_\mathfrak{p}(\mathfrak{t})=1\\ \chi_\mathfrak{p}^2(\varpi_\mathfrak{p})=-1}}\mathfrak{p}\prod_{\substack{\mathfrak{p}|\mathfrak{t},\,\mathfrak{p}\nmid\mathfrak{c}_\chi\\ v_\mathfrak{p}(\mathfrak{t})\geqslant 3}}\mathfrak{p}^{v_\mathfrak{p}(\mathfrak{t})-2}\prod_{\mathfrak{p}|\mathfrak{t},\,\mathfrak{p}|\mathfrak{c}_\chi}\mathfrak{p}^{v_\mathfrak{p}(\mathfrak{t})}, \tag{61}$$

and

$$\left|\rho_{E(\varphi)}(t_\chi)\right| = \frac{\pi^{d/2}|D_K|^{-1/2}}{\left|L^{(\mathfrak{t}\mathfrak{t}_\chi^{-1})}(1,\chi^2)\right|(\mathcal{N}\mathfrak{t}\mathfrak{t}_\chi^{-1})^{1/2}F_{\chi,\mathfrak{t}}}\,, \tag{62}$$

where $L^{(\mathfrak{t}\mathfrak{t}_\chi^{-1})}(\cdot,\chi^2)$ denotes a partial Hecke L-function (note that $L(s,\chi^2)$ is holomorphic and nonzero at $s = 1$, because χ^2 is a nontrivial Hecke character), and

$$F_{\chi,\mathfrak{t}} := \prod_{\substack{\mathfrak{p}|\mathfrak{t},\,\mathfrak{p}\nmid\mathfrak{c}_\chi\\ v_\mathfrak{p}(\mathfrak{t})=1\\ \chi_\mathfrak{p}^2(\varpi_\mathfrak{p})\neq-1}}|1+\chi_\mathfrak{p}^2(\varpi_\mathfrak{p})|^{-1}. \tag{63}$$

For convenience we note that

$$\mathfrak{t}\mathfrak{t}_\chi^{-1} = \prod_{\substack{\mathfrak{p}|\mathfrak{t},\,\mathfrak{p}\nmid\mathfrak{c}_\chi\\ v_\mathfrak{p}(\mathfrak{t})=1\\ \chi_\mathfrak{p}^2(\varpi_\mathfrak{p})\neq-1}}\mathfrak{p}\prod_{\substack{\mathfrak{p}|\mathfrak{t},\,\mathfrak{p}\nmid\mathfrak{c}_\chi\\ v_\mathfrak{p}(\mathfrak{t})\geqslant 2}}\mathfrak{p}^2,$$

so that for $\mathfrak{p} \nmid \mathfrak{c}_\chi$ the relation $\mathfrak{p} \nmid \mathfrak{t}\mathfrak{t}_\chi^{-1}$ is equivalent to $\mathfrak{p} \mid \mathfrak{t}$, $v_\mathfrak{p}(\mathfrak{t}) = 1$, $\chi_\mathfrak{p}^2(\varpi_\mathfrak{p}) = -1$. The coefficients $\rho_{E(\varphi)}(\mathfrak{m})$ enjoy the property

$$\rho_{E(\varphi)}(\mathfrak{m}\mathfrak{t}_\chi) = \frac{\lambda_{\chi,\mathfrak{t}}(\mathfrak{m})}{\sqrt{\mathcal{N}\mathfrak{m}}}\rho_{E(\varphi)}(\mathfrak{t}_\chi)\,, \quad \mathfrak{m} \subseteq \mathfrak{o}\,, \tag{64}$$

where $\lambda_{\chi,\mathfrak{t}}$ is a multiplicative function on nonzero integral ideals satisfying the identity

$$\lambda_{\chi,\mathfrak{t}}(\mathfrak{m}) = \begin{cases} \sum_{\mathfrak{a}\mathfrak{b}=\mathfrak{m}} \chi(\mathfrak{a}\mathfrak{b}^{-1}), & \gcd(\mathfrak{m}, \mathfrak{t}\mathfrak{t}_\chi^{-1}\mathfrak{c}_\chi) = \mathfrak{o}, \\ 0, & \gcd(\mathfrak{m}, \mathfrak{c}_\chi) \neq \mathfrak{o}, \end{cases}$$

and the general bound

$$\begin{aligned} |\lambda_{\chi,\mathfrak{t}}(\mathfrak{m})| &\leqslant \tau(\mathfrak{m}) \prod_{\substack{\mathfrak{p}|\mathfrak{t},\, \mathfrak{p}|\mathfrak{m},\, \mathfrak{p}\nmid\mathfrak{c}_\chi \\ v_\mathfrak{p}(\mathfrak{t})=1 \\ \chi_\mathfrak{p}^2(\varpi_\mathfrak{p})\neq -1}} \frac{\mathcal{N}\mathfrak{p}}{|1+\chi_\mathfrak{p}^2(\varpi_\mathfrak{p})|} \prod_{\substack{\mathfrak{p}|\mathfrak{t},\, \mathfrak{p}|\mathfrak{m},\, \mathfrak{p}\nmid\mathfrak{c}_\chi \\ v_\mathfrak{p}(\mathfrak{t})\geqslant 2}} (\mathcal{N}\mathfrak{p})^2 \\ &\leqslant F_{\chi,\mathfrak{t}}\tau(\mathfrak{t})\big(\mathcal{N}(\mathfrak{t}\mathfrak{t}_\chi^{-1})\big)^{1/2}\big(\mathcal{N}\gcd(\mathfrak{t}\mathfrak{t}_\chi^{-1},\mathfrak{m})\big)\tau(\mathfrak{m}). \end{aligned} \tag{65}$$

We are now ready to write down explicitly the Fourier expansion (30) for our specific $\phi = E(\varphi) \in R^{(\mathfrak{t})}V_{\chi,\chi^{-1}}(\mathfrak{c}_\chi^2)$ introduced at the beginning of this section. We have

$$\rho_{E(\varphi),0}(y) = \varphi\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) + \int_{\mathbb{A}} \varphi\left(\begin{pmatrix} 0 & -1 \\ y & \xi \end{pmatrix}\right) d\xi, \quad y \in \mathbb{A}^\times, \tag{66}$$

and

$$\varepsilon_{\chi,\chi^{-1}}(\mathrm{sgn}(y)) = \chi_\infty(\mathrm{sgn}(y)), \quad y \in K_\infty^\times. \tag{67}$$

With the notation (61), (63) and (64) we define

$$\lambda^{(\mathfrak{t})}_{\chi,\chi^{-1}}(\mathfrak{m}) := \begin{cases} F_{\chi,\mathfrak{t}}^{-1}\tau(\mathfrak{t})^{-1}(\mathcal{N}\mathfrak{t})^{-1/2}(\mathcal{N}\mathfrak{t}_\chi)\lambda_{\chi,\mathfrak{t}}(\mathfrak{m}t_\chi^{-1}), & \mathfrak{t}_\chi \mid \mathfrak{m}, \\ 0, & \text{otherwise}, \end{cases} \tag{68}$$

and

$$W_{E(\varphi)}(y) := F_{\chi,\mathfrak{t}}\tau(\mathfrak{t})\big(\mathcal{N}(\mathfrak{t}\mathfrak{t}_\chi^{-1})\big)^{1/2}\rho_{E(\varphi)}(t_\chi)\varepsilon_{\chi,\chi^{-1}}(\mathrm{sgn}(y))\tilde{W}_{q/2,\nu_{\chi,\chi^{-1}}}(y), \quad y \in K_\infty^\times. \tag{69}$$

Then (48) follows from (30), (64), (68), (69); (49) follows in slightly sharper form from (65), (68); (50) follows from (7), (13), (25), (51), (62), (69).

It is worthwhile to review the special case when $\mathfrak{c} = \mathfrak{c}_\chi^2$. Then $\mathfrak{t} = \mathfrak{t}_\chi = \mathfrak{o}$ and $\phi = E(\varphi)$ spans the space $V_{\chi,\chi^{-1},q}(\mathfrak{c}_\chi^2)$ of newforms of pure weight q. We can define W_ϕ intrinsically by (36), and the coefficients $\lambda^{(\mathfrak{t})}_{\chi,\chi^{-1}}$ in this case specialize to the Hecke eigenvalues given for $\mathfrak{m} \subseteq \mathfrak{o}$ by

$$\lambda_{\chi,\chi^{-1}}(\mathfrak{m}) = \begin{cases} \sum_{\mathfrak{a}\mathfrak{b}=\mathfrak{m}} \chi(\mathfrak{a}\mathfrak{b}^{-1}), & \gcd(\mathfrak{m}, \mathfrak{c}_\chi) = \mathfrak{o}, \\ 0, & \text{otherwise}. \end{cases}$$

By (25) and the above discussion, newforms $\phi_1, \phi_2 \in V_{\chi,\chi^{-1}}(\mathfrak{c}_\chi^2)$ satisfy the analogue of (37),

$$\langle \phi_1, \phi_2 \rangle = C_{\chi,\chi^{-1}} \langle W_{\phi_1}, W_{\phi_2} \rangle$$

with some positive constant $C_{\chi,\chi^{-1}} \gg_{K,\varepsilon} C(\chi,\chi^{-1})^{-\varepsilon}$ depending only on χ.

2.8 The constant term of a certain Eisenstein series. For a Rankin–Selberg type computation in the next section we need to understand in more detail the constant term of a certain Eisenstein series. For $s \in \mathbb{C}$ let χ_1, χ_2 be the quasicharacters defined by $\chi_1(y) := |y|^s$, $\chi_2(y) := |y|^{-s}$ for $y \in \mathbb{A}^\times$. For a nonzero ideal

$\mathfrak{c} \subseteq \mathfrak{o}$ let us define $\varphi = \varphi(s, g) \in H(\chi_1, \chi_2)$ by

$$\varphi\left(s, \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} k\right) := \begin{cases} \left|\frac{a}{b}\right|^{1/2+s}, & k \in \mathrm{SO}_2(K_\infty) \times \mathcal{K}(\mathfrak{c}), \\ 0, & k \in \mathcal{K} - \big(\mathrm{SO}_2(K_\infty) \times \mathcal{K}(\mathfrak{c})\big). \end{cases} \tag{70}$$

The constant term of $E(\varphi(s), g)$ equals (cf. (66) and [GJ, (5.3)])

$$E_{\mathrm{const}}\big(\varphi(s), g\big) := \varphi(s, g) + \int_{\mathbb{A}} \varphi\left(s, \begin{pmatrix} 0 & -1 \\ 1 & \xi \end{pmatrix} g\right) d\xi, \quad g \in \mathrm{GL}_2(\mathbb{A}). \tag{71}$$

The aim of this section is to prove that

$$\int_{\mathbb{A}} \varphi\left(s, \begin{pmatrix} 0 & -1 \\ 1 & \xi \end{pmatrix} g\right) d\xi = \frac{\Lambda_K(2s)}{\Lambda_K(1+2s)} H(s, g), \tag{72}$$

where

$$\Lambda_K(s) := |D_K|^{s/2}\big(\pi^{-s/2}\Gamma(s/2)\big)^d \prod_{\mathfrak{p}} \big(1 - (\mathcal{N}\mathfrak{p})^{-s}\big)^{-1}, \quad \Re s > 1,$$

is the completed Dedekind zeta-function, and $H(s, g)$ is a meromorphic function whose zeros and poles lie on $\Re s = 0$ and $\Re s = -1/2$, respectively, and which is constant at $s = 1/2$:

$$H(1/2, g) = |\delta|(\mathcal{N}\mathfrak{c})^{-1} \prod_{\mathfrak{p}|\mathfrak{c}} \big(1 + (\mathcal{N}\mathfrak{p})^{-1}\big)^{-1} = |D_K|^{-1}\big[\mathcal{K}(\mathfrak{o}) : \mathcal{K}(\mathfrak{c})\big]^{-1}.$$

This is a general feature (cf. [Bu, Prop. 3.7.5] and [GJ, p. 277]), but we preferred to prove it by direct calculation.

In order to understand the integral in (71), we define $\delta \in \mathbb{A}^\times$ as in the previous section, and then using the Iwasawa and Bruhat decompositions we write

$$g = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} h, \quad x \in \mathbb{A},\ a, b \in \mathbb{A}^\times,\ h \in \mathrm{GL}_2(\mathbb{A}),$$

where $h_\infty \in \mathrm{SO}_2(K_\infty)$, $h_\mathfrak{p} \in \mathcal{K}(\mathfrak{o}_\mathfrak{p})$ for $\mathfrak{p} \nmid \mathfrak{c}$, and $h_\mathfrak{p} \in \mathrm{GL}_2(K_\mathfrak{p})$ is either $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or of the form $\left(\begin{smallmatrix} 0 & -\delta_\mathfrak{p}^{-1} \\ \delta_\mathfrak{p} & \eta_\mathfrak{p} \end{smallmatrix}\right)$ for $\mathfrak{p} \mid \mathfrak{c}$. After simple manipulations the integral in (71) becomes

$$\left|\frac{a}{b}\right|^{1/2-s} |\delta|^{2s} \int_{\mathbb{A}} \varphi\left(s, \begin{pmatrix} 0 & -\delta^{-1} \\ \delta & \xi \end{pmatrix} h\right) d\xi.$$

The new integral decomposes as a product of local factors in a natural fashion, using that φ is the tensor product of its restrictions φ_∞ and $\varphi_\mathfrak{p}$ to the various quasifactors of $\mathrm{GL}_2(\mathbb{A})$.

The infinite part of the new integral equals, by our choice of the Haar measure on K_∞, the right $\mathrm{SO}_2(K_\infty)$-invariance of $\varphi_\infty(s, \cdot)$, the Iwasawa decomposition (54), and the formula [GrR, 3.251.2],

$$|D_K|^{-1/2}\left(\int_{-\infty}^{\infty} \frac{d\xi}{(\xi^2+1)^{1/2+s}}\right)^d = |D_K|^{-1/2}\left(\frac{\Gamma(1/2)\Gamma(s)}{\Gamma(1/2+s)}\right)^d.$$

We now calculate the local factor corresponding to a prime $\mathfrak{p}$. For simplicity we shall omit the subscripts in $\delta_\mathfrak{p}$ and $\eta_\mathfrak{p}$. If $\mathfrak{p} \nmid \mathfrak{c}$, then $\varphi_\mathfrak{p}(s, \cdot)$ is right $\mathcal{K}(\mathfrak{o}_\mathfrak{p})$-invariant, hence by the Iwasawa decomposition (56) the local factor corresponding to $\mathfrak{p}$ equals

$$\int_{K_\mathfrak{p}} \varphi_\mathfrak{p}\left(s, \begin{pmatrix} 0 & -\delta^{-1} \\ \delta & \xi \end{pmatrix}\right) d\xi = \int_{v_\mathfrak{p}(\xi) \geqslant 0} 1\, d\xi + \int_{v_\mathfrak{p}(\xi) < 0} |\xi|^{-1-2s}\, d\xi$$

$$= 1 + \sum_{j=1}^{\infty} (\mathcal{N}\mathfrak{p})^{-j(1+2s)} (\mathcal{N}\mathfrak{p})^{j} \left(1 - \frac{1}{\mathcal{N}\mathfrak{p}}\right)$$
$$= \frac{1 - (\mathcal{N}\mathfrak{p})^{-1-2s}}{1 - (\mathcal{N}\mathfrak{p})^{-2s}} .$$

If $\mathfrak{p} \mid \mathfrak{c}$ then depending on the shape of $h_{\mathfrak{p}}$ the local factor is either

$$\int_{K_{\mathfrak{p}}} \varphi_{\mathfrak{p}} \left(s, \begin{pmatrix} 0 & -\delta^{-1} \\ \delta & \xi \end{pmatrix}\right) d\xi \tag{73}$$

or

$$\int_{K_{\mathfrak{p}}} \varphi_{\mathfrak{p}} \left(s, \begin{pmatrix} 0 & -\delta^{-1} \\ \delta & \xi \end{pmatrix} \begin{pmatrix} 0 & -\delta^{-1} \\ \delta & \eta \end{pmatrix}\right) d\xi . \tag{74}$$

The integral (73) equals, by the Iwasawa decomposition (56),

$$\int_{K_{\mathfrak{p}}} \varphi_{\mathfrak{p}} \left(s, \begin{pmatrix} 0 & -\delta^{-1} \\ \delta & \xi \end{pmatrix}\right) d\xi = \int_{v_{\mathfrak{p}}(\xi) \leqslant -v_{\mathfrak{p}}(\mathfrak{c})} |\xi|^{-1-2s} \, d\xi$$
$$= \sum_{j=v_{\mathfrak{p}}(\mathfrak{c})}^{\infty} (\mathcal{N}\mathfrak{p})^{-j(1+2s)} (\mathcal{N}\mathfrak{p})^{j} \left(1 - \frac{1}{\mathcal{N}\mathfrak{p}}\right)$$
$$= (\mathcal{N}\mathfrak{p})^{-(2s)v_{\mathfrak{p}}(\mathfrak{c})} \frac{1 - (\mathcal{N}\mathfrak{p})^{-1}}{1 - (\mathcal{N}\mathfrak{p})^{-2s}} .$$

If $v_{\mathfrak{p}}(\eta) \leqslant -v_{\mathfrak{p}}(\mathfrak{c})$, then by the right $\mathcal{K}(\mathfrak{c}_{\mathfrak{p}})$ invariance of $\varphi_{\mathfrak{p}}(s, \cdot)$ the integral (74) equals

$$\int_{K_{\mathfrak{p}}} \varphi_{\mathfrak{p}} \left(s, \begin{pmatrix} 0 & -\delta^{-1} \\ \delta & \xi \end{pmatrix} \begin{pmatrix} 0 & -\delta^{-1} \\ \delta & \eta \end{pmatrix}\right) d\xi = \int_{K_{\mathfrak{p}}} \varphi_{\mathfrak{p}} \left(s, \begin{pmatrix} 0 & -\delta^{-1}\eta \\ \delta\eta^{-1} & \eta\xi - 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \delta\eta^{-1} & 1 \end{pmatrix}\right) d\xi$$
$$= |\eta|^{1+2s} \int_{K_{\mathfrak{p}}} \varphi_{\mathfrak{p}} \left(s, \begin{pmatrix} 0 & -\delta^{-1} \\ \delta & \eta^2\xi - \eta \end{pmatrix}\right) d\xi$$
$$= |\eta|^{2s-1} (\mathcal{N}\mathfrak{p})^{-(2s)v_{\mathfrak{p}}(\mathfrak{c})} \frac{1 - (\mathcal{N}\mathfrak{p})^{-1}}{1 - (\mathcal{N}\mathfrak{p})^{-2s}} ,$$

where in the last step we combined the change of variable $\xi \to \eta^{-2}(\xi + \eta)$ with our previous result for (73). If $v_{\mathfrak{p}}(\eta) > -v_{\mathfrak{p}}(\mathfrak{c})$, then (74) equals

$$\int_{K_{\mathfrak{p}}} \varphi_{\mathfrak{p}} \left(s, \begin{pmatrix} 0 & -\delta^{-1} \\ \delta & \xi \end{pmatrix} \begin{pmatrix} 0 & -\delta^{-1} \\ \delta & \eta \end{pmatrix}\right) d\xi = \int_{K_{\mathfrak{p}}} \varphi_{\mathfrak{p}} \left(s, \begin{pmatrix} -1 & -\delta^{-1}\eta \\ \delta\xi & \eta\xi - 1 \end{pmatrix}\right) d\xi$$
$$= \int_{K_{\mathfrak{p}}} \varphi_{\mathfrak{p}} \left(s, \begin{pmatrix} 0 & -\delta^{-1}\xi^{-1} \\ \delta\xi & \eta\xi - 1 \end{pmatrix}\right) d\xi$$
$$= \int_{K_{\mathfrak{p}}} |\xi|^{-1-2s} \varphi_{\mathfrak{p}} \left(s, \begin{pmatrix} 0 & -\delta^{-1} \\ \delta & \eta - \xi^{-1} \end{pmatrix}\right) d\xi .$$

Using the change of variable $\xi \to (\eta - \xi)^{-1}$ and the Iwasawa decomposition (56) we obtain

$$\int_{K_{\mathfrak{p}}} \varphi_{\mathfrak{p}} \left(s, \begin{pmatrix} 0 & -\delta^{-1} \\ \delta & \xi \end{pmatrix} \begin{pmatrix} 0 & -\delta^{-1} \\ \delta & \eta \end{pmatrix}\right) d\xi = \int_{K_{\mathfrak{p}}} |\eta - \xi|^{2s-1} \varphi_{\mathfrak{p}} \left(s, \begin{pmatrix} 0 & -\delta^{-1} \\ \delta & \xi \end{pmatrix}\right) d\xi$$

$$= \int_{v_\mathfrak{p}(\xi)\leqslant -v_\mathfrak{p}(\mathfrak{c})} |\eta-\xi|^{2s-1}|\xi|^{-1-2s}d\xi\,.$$

By $v_\mathfrak{p}(\xi) \leqslant -v_\mathfrak{p}(\mathfrak{c}) < v_\mathfrak{p}(\eta)$ we have $|\eta-\xi| = |\xi|$, hence by the change of variable $\xi \to \xi^{-1}$

$$\int_{K_\mathfrak{p}} \varphi_\mathfrak{p}\left(s, \begin{pmatrix} 0 & -\delta^{-1} \\ \delta & \xi \end{pmatrix} \begin{pmatrix} 0 & -\delta^{-1} \\ \delta & \eta \end{pmatrix}\right) d\xi = \int_{v_\mathfrak{p}(\xi)\leqslant -v_\mathfrak{p}(\mathfrak{c})} \frac{d\xi}{|\xi|^2}$$
$$= \int_{v_\mathfrak{p}(\xi)\geqslant v_\mathfrak{p}(\mathfrak{c})} 1\, d\xi = (\mathcal{N}\mathfrak{p})^{-v_\mathfrak{p}(\mathfrak{c})}.$$

This equation is also true for $\eta = 0$, because then $|\eta - \xi| = |\xi|$ holds trivially. Collecting the previous computations, we arrive at (72).

2.9 L-functions on $\mathbf{GL_2}$ and $\mathbf{GL_2 \times GL_2}$. Let (π, V_π) be an irreducible cuspidal representation. Let χ be a Hecke character of conductor $\mathfrak{q}$. The twisted L-function

$$L(s, \pi\otimes\chi) = \sum_{\mathfrak{m}} \lambda_{\pi\otimes\chi}(\mathfrak{m})(N\mathfrak{m})^{-s}, \quad \Re s > 1\,,$$

can be continued to an entire function on $\mathbb{C}$ and satisfies a functional equation relating s to $1-s$ with analytic conductor (see (13))

$$C\big(\pi\otimes\chi\otimes|\det|^{s-1/2}\big) \asymp_{\pi,\chi_\infty} (\mathcal{N}\mathfrak{q})^2\big(1+|\Im s|\big)^2.$$

The Hecke eigenvalues $\lambda_{\pi\otimes\chi}(\mathfrak{m})$ satisfy the bound (33) and the identity

$$\lambda_{\pi\otimes\chi}(\mathfrak{m}) = \lambda_\pi(\mathfrak{m})\chi(\mathfrak{m})\,, \quad \gcd(\mathfrak{m}, \mathfrak{q}\mathfrak{c}_\pi) = \mathfrak{o}\,.$$

By [H, Th. 2.1] we can express $L(1/2, \pi\otimes\chi)$ as an essentially finite series: there is a complex number η of modulus 1 and a smooth function $V : (0,\infty) \to \mathbb{C}$ with rapidly decaying derivatives, both depending only on the archimedean parameters of $\pi\otimes\chi$, such that

$$L(1/2, \pi\otimes\chi) = \Sigma + \eta\bar{\Sigma}\,, \quad \Sigma := \sum_{\{0\}\neq\mathfrak{m}\subseteq\mathfrak{o}} \frac{\lambda_{\pi\otimes\chi}(\mathfrak{m})}{\sqrt{\mathcal{N}\mathfrak{m}}} V\left(\frac{\mathcal{N}\mathfrak{m}}{\sqrt{C(\pi\otimes\chi)}}\right).$$

Together with a smooth partition of unity and standard bounds for $\lambda_{\pi\otimes\chi}$ at ramified primes we obtain (cf. e.g. [BlHM, §5.1])

$$L(1/2, \pi\otimes\chi) \ll_{\pi,\chi_\infty,\varepsilon} (\mathcal{N}\mathfrak{q})^\varepsilon \max_{Y\leqslant c(\mathcal{N}\mathfrak{q})^{1+\varepsilon}} \left| \sum_{\{0\}\neq\mathfrak{m}\subseteq\mathfrak{o}} \frac{\lambda_\pi(\mathfrak{m})\chi(\mathfrak{m})}{\sqrt{\mathcal{N}\mathfrak{m}}} V\left(\frac{\mathcal{N}\mathfrak{m}}{Y}\right)\right|, \tag{75}$$

where $c = c(\pi, \chi_\infty, \varepsilon) > 0$ is a constant and $V : (0,\infty) \to \mathbb{C}$ is a smooth function supported on $[1/2, 2]$ such that $V^{(j)}(y) \ll_{\pi,\chi_\infty,j} 1$ for all for all $j \in \mathbb{N}_0$.

The Rankin–Selberg convolution $L(s, \pi\otimes\tilde{\pi})$ (with $\tilde{\pi}$ the contragradient representation) is, up to finitely many Euler factors at the primes dividing $\mathfrak{c}_\pi$, given by

$$\zeta_K(2s) \sum_{\{0\}\neq\mathfrak{m}\subseteq\mathfrak{o}} \frac{|\lambda_\pi(\mathfrak{m})|^2}{(\mathcal{N}\mathfrak{m})^s}\,.$$

It is an entire function of order 1 except for a simple pole at $s = 1$ and satisfies a functional equation relating s to $1-s$, see e.g. [HoR, Rem. 1.2], and the references given there. By [HoR, Lem. b], the analytic conductor satisfies

$$C(\pi)^{-B} \ll C(\pi\otimes\tilde{\pi}) \ll C(\pi)^{B} \tag{76}$$

for some constant B depending only on K. By standard contour integration we obtain

$$\sum_{\mathcal{N}\mathfrak{m}\leqslant x} \left|\lambda_\pi(\mathfrak{m})\right|^2 \ll C(\pi)^{B'}x \tag{77}$$

for $x\geqslant 1$ and some constant B' depending only on K.

Let $\phi_1,\phi_2\in V_{\pi,q}(\mathfrak{c}_\pi)$ be newforms of some weight $q\in\mathbb{Z}^d$ and let $\mathfrak{t}_1,\mathfrak{t}_2\subseteq\mathfrak{o}$ be nonzero ideals. For any integral ideal $\mathfrak{c}$ divisible by $\mathfrak{t}_1\mathfrak{c}_\pi$, $\mathfrak{t}_2\mathfrak{c}_\pi$ the vectors $\psi_i:=R_{\mathfrak{t}_i}\phi_i$ lie in $V_{\pi,q}(\mathfrak{c})$ (cf. (17)), and our aim is to express their inner product in terms of the inner product of the Whittaker functions W_{ϕ_i}. For this purpose we shall apply the Rankin–Selberg unfolding technique to the function

$$F(s):=\int_{\mathrm{GL}_2(K)Z(\mathbb{A})\backslash\mathrm{GL}_2(\mathbb{A})}\psi_1(g)\bar{\psi}_2(g)E\big(\varphi(s),g\big)\,dg\,,$$

where $\varphi(s,g)$ is defined by (70). It is known from the theory of Eisenstein series, that $E(\varphi(s),g)$ is meromorphic in s with all the singularities coming from the constant term $E_{\mathrm{const}}(\varphi(s),g)$, more precisely from the integral in (71), see [GJ, §5]. The result of the previous section shows that $E(\varphi(s),g)$ has a pole at $s=1/2$ with constant residue

$$\operatorname*{res}_{s=1/2}E\big(\varphi(s),g\big)=\frac{C_K}{[\mathcal{K}(\mathfrak{o}):\mathcal{K}(\mathfrak{c})]}\,,\quad C_K:=\frac{\mathrm{res}_{s=1}\Lambda_K(s)}{2|D_K|\Lambda_K(2)}\,.$$

In particular,

$$\operatorname*{res}_{s=1/2}F(s)=C_K\frac{\langle R_{\mathfrak{t}_1}\phi_1,R_{\mathfrak{t}_1}\phi_2\rangle}{[\mathcal{K}(\mathfrak{o}):\mathcal{K}(\mathfrak{c})]}\,. \tag{78}$$

On the other hand, unfolding the integral we see (cf. [KniL1, Prop. 7.47] or [Bu, p. 372–373]) that

$$\begin{aligned}F(s)&=\int_{P(K)Z(\mathbb{A})\backslash\mathrm{GL}_2(\mathbb{A})}\psi_1(g)\bar{\psi}_2(g)\varphi(s,g)dg\\&=\int_{K^\times\backslash\mathbb{A}^\times}\int_{K\backslash\mathbb{A}}\int_{\mathcal{K}}\psi_1\left(\begin{pmatrix}y&x\\0&1\end{pmatrix}k\right)\bar{\psi}_2\left(\begin{pmatrix}y&x\\0&1\end{pmatrix}k\right)\varphi\left(s,\begin{pmatrix}y&x\\0&1\end{pmatrix}k\right)dk\,dx\,\frac{d^\times y}{|y|}\\&=\int_{K^\times\backslash\mathbb{A}^\times}\int_{K\backslash\mathbb{A}}\int_{\mathrm{SO}_2(K_\infty)\times\mathcal{K}(\mathfrak{c})}\psi_1\left(\begin{pmatrix}y&x\\0&1\end{pmatrix}k\right)\bar{\psi}_2\left(\begin{pmatrix}y&x\\0&1\end{pmatrix}k\right)|y|^{s-\frac12}dk\,dx\,d^\times y\,.\end{aligned}$$

Since $\psi_1\bar{\psi}_2$ is right $\mathrm{SO}_2(K_\infty)\times\mathcal{K}(\mathfrak{c})$-invariant, we obtain

$$F(s)=\frac{1}{[\mathcal{K}(\mathfrak{o}):\mathcal{K}(\mathfrak{c})]}\int_{K^\times\backslash\mathbb{A}^\times}\int_{K\backslash\mathbb{A}}\psi_1\left(\begin{pmatrix}y&x\\0&1\end{pmatrix}\right)\bar{\psi}_2\left(\begin{pmatrix}y&x\\0&1\end{pmatrix}\right)|y|^{s-\frac12}dx\,d^\times y\,.$$

We choose any finite ideles $t_i\in\mathbb{A}_{\mathrm{fin}}^\times$ representing the ideals $\mathfrak{t}_i$, so that $\psi_i(g)=\phi_i\left(g\left(\begin{smallmatrix}t_i^{-1}&0\\0&1\end{smallmatrix}\right)\right)$. We insert the Fourier expansion (34), and integrate over x getting

$$\begin{aligned}F(s)=\frac{(\mathcal{N}\mathfrak{t}_1\mathfrak{t}_2)^{1/2}}{[\mathcal{K}(\mathfrak{o}):\mathcal{K}(\mathfrak{c})]}\int_{K^\times\backslash\mathbb{A}^\times}\sum_{r\in K^\times}&\frac{\lambda_\pi(ry_{\mathrm{fin}}t_1^{-1})\bar{\lambda}_\pi(ry_{\mathrm{fin}}t_1^{-1})}{\mathcal{N}(ry_{\mathrm{fin}})}\\&W_{\phi_1}(ry_\infty)\bar{W}_{\phi_2}(ry_\infty)|y|^{s-\frac12}d^\times y\end{aligned}$$

$$= \frac{(\mathcal{N}\mathfrak{t}_1\mathfrak{t}_2)^{1/2}}{[\mathcal{K}(\mathfrak{o}):\mathcal{K}(\mathfrak{c})]}\int_{\mathbb{A}^\times}\frac{\lambda_\pi(y_{\mathrm{fin}}t_1^{-1})\bar{\lambda}_\pi(y_{\mathrm{fin}}t_1^{-1})}{\mathcal{N}(y_{\mathrm{fin}})}W_{\phi_1}(y_\infty)\bar{W}_{\phi_2}(y_\infty)|y|^{s-\frac{1}{2}}\,d^\times y,$$
$$= \frac{(\mathcal{N}\mathfrak{t}_1\mathfrak{t}_2)^{1/2}}{[\mathcal{K}(\mathfrak{o}):\mathcal{K}(\mathfrak{c})]}\left(\int_{K_\infty^\times}W_{\phi_1}(y)\bar{W}_{\phi_2}(y)|y|^{s-\frac{1}{2}}d^\times y\right)\left(\int_{\mathbb{A}_{\mathrm{fin}}^\times}\frac{\lambda_\pi(yt_1^{-1})\bar{\lambda}_\pi(yt_2^{-1})}{(\mathcal{N}(y))^{\frac{1}{2}+s}}\,d^\times y\right).$$

We choose any $t\in\mathbb{A}_{\mathrm{fin}}^\times$ representing the ideal $\gcd(\mathfrak{t}_1,\mathfrak{t}_2)$, and we make the change of variable $y\to yt_1t_2t^{-1}$ in the second integral. We obtain

$$F(s)=\frac{(\mathcal{N}\gcd(\mathfrak{t}_1,\mathfrak{t}_2))^{\frac{1}{2}+s}}{(\mathcal{N}\mathfrak{t}_1\mathfrak{t}_2)^s[\mathcal{K}(\mathfrak{o}):\mathcal{K}(\mathfrak{c})]}$$
$$\left(\int_{K_\infty^\times}W_{\phi_1}(y)\bar{W}_{\phi_2}(y)|y|^{s-\frac{1}{2}}\,d^\times y\right)\left(\int_{\mathbb{A}_{\mathrm{fin}}^\times}\frac{\lambda_\pi(yt_2')\bar{\lambda}_\pi(yt_1')}{(\mathcal{N}(y))^{\frac{1}{2}+s}}\,d^\times y\right),$$

where the finite ideles $t_i':=t_it^{-1}$ represent the coprime integral ideals $\mathfrak{t}_i:=\mathfrak{t}_i\gcd(\mathfrak{t}_1,\mathfrak{t}_2)^{-1}$. We conclude

$$\operatorname*{res}_{s=\frac{1}{2}}F(s)=\frac{\langle W_{\phi_1},W_{\phi_2}\rangle}{(\mathcal{N}\mathfrak{t}_1'\mathfrak{t}_2')^{1/2}[\mathcal{K}(\mathfrak{o}):\mathcal{K}(\mathfrak{c})]}\operatorname*{res}_{s=1}\sum_{\{0\}\neq\mathfrak{m}\subseteq\mathfrak{o}}\frac{\lambda_\pi(\mathfrak{m}\mathfrak{t}_2')\bar{\lambda}_\pi(\mathfrak{m}\mathfrak{t}_1')}{(\mathcal{N}\mathfrak{m})^s},$$

whence by (78) also

$$\langle R_{\mathfrak{t}_1}\phi_1,R_{\mathfrak{t}_1}\phi_2\rangle=\frac{\langle W_{\phi_1},W_{\phi_2}\rangle}{C_K(\mathcal{N}\mathfrak{t}_1'\mathfrak{t}_2')^{1/2}}\operatorname*{res}_{s=1}\sum_{\{0\}\neq\mathfrak{m}\subseteq\mathfrak{o}}\frac{\lambda_\pi(\mathfrak{m}\mathfrak{t}_2')\bar{\lambda}_\pi(\mathfrak{m}\mathfrak{t}_1')}{(\mathcal{N}\mathfrak{m})^s}.$$

We now draw some useful consequences of this identity. First, combining the special case $\mathfrak{t}_1=\mathfrak{t}_2=\mathfrak{o}$ with (16), (25), and taking into account the ramified primes, we arrive at (37) with some positive constant C_π depending only on π which satisfies

$$(\mathcal{N}\mathfrak{c}_\pi)^{-\varepsilon}\operatorname*{res}_{s=1}L(s,\pi\otimes\tilde{\pi})\ll_{K,\varepsilon}C_\pi\ll_{K,\varepsilon}(\mathcal{N}\mathfrak{c}_\pi)^{\varepsilon}\operatorname*{res}_{s=1}L(s,\pi\otimes\tilde{\pi}).\tag{79}$$

Second, comparing the special case with the general one, we infer (cf. [ILS, p. 73])

$$\langle R_{\mathfrak{t}_1}\phi_1,R_{\mathfrak{t}_2}\phi_2\rangle=\frac{\langle\phi_1,\phi_2\rangle}{(\mathcal{N}\mathfrak{t}_1'\mathfrak{t}_2')^{1/2}}\prod_{\mathfrak{p}^{\nu_1}\|\mathfrak{t}_1'}\left(\sum_{k=0}^\infty\frac{\lambda_\pi(\mathfrak{p}^k)\bar{\lambda}_\pi(\mathfrak{p}^{k+\nu_1})}{(\mathcal{N}\mathfrak{p})^k}\right)\left(\sum_{k=0}^\infty\frac{|\lambda_\pi(\mathfrak{p}^k)|^2}{(\mathcal{N}\mathfrak{p})^k}\right)^{-1}$$
$$\prod_{\mathfrak{p}^{\nu_2}\|\mathfrak{t}_2'}\left(\sum_{k=0}^\infty\frac{\lambda_\pi(\mathfrak{p}^{k+\nu_2})\bar{\lambda}_\pi(\mathfrak{p}^k)}{(\mathcal{N}\mathfrak{p})^k}\right)\left(\sum_{k=0}^\infty\frac{|\lambda_\pi(\mathfrak{p}^k)|^2}{(\mathcal{N}\mathfrak{p})^k}\right)^{-1}.\tag{80}$$

An important feature here is that the ratio of the two inner products is independent of the weight $q\in\mathbb{Z}^d$. This independence is key to the existence of the operators (38); it can also be verified directly by using the Maaß shift operators.

We now are ready to prove the following lemma.

LEMMA 3. *Let (π,V_π) be an irreducible cuspidal representation of $\mathrm{GL}_2(K)\backslash\mathrm{GL}_2(\mathbb{A})$ with unitary central character, and let $\phi\in V_\pi(\mathfrak{c}_\pi)$. Then*

$$C(\pi)^{-\varepsilon}\|\phi\|\ll_{K,\varepsilon}\|W_\phi\|\ll_{K,\varepsilon}C(\pi)^{\varepsilon}\|\phi\|.$$

The implied constants are ineffective and depend only on K and ε.

Proof. By (37) and (79) it remains to show

$$C(\pi)^{-\varepsilon} \ll_{K,\varepsilon} \operatorname*{res}_{s=1} L(s,\pi\otimes\tilde{\pi}) \ll_{K,\varepsilon} C(\pi)^{\varepsilon}.$$

For $K=\mathbb{Q}$ this is known from the work of Iwaniec [I2, Th. 2] (upper bound) and Hoffstein–Lockhart [HoL] (lower bound). The same bounds are also available in the number field case and can be obtained as follows: the upper bound follows verbatim as in [I2, p. 72–73, especially the comment between (20) and (21)] once we have the multiplicativity relation (31), and we know that $L(s,\pi\otimes\tilde{\pi})$ is of order 1 and holomorphic except for a simple pole at $s=1$, and satisfies a suitable functional equation with conductor satisfying (76). For the lower bound, [HoL, Prop. 1] together with (76) gives the desired bound. Note that $L(s,\pi\otimes\tilde{\pi})$ has nonnegative coefficients (which incidentally holds by [HoR, Lem. a] in a much more general context). To verify the hypothesis "no Siegel zeros" for the application of this result, we distinguish between two cases depending on whether $\mathrm{ad}^2\pi$ is cuspidal or not. In the first case, the absence of Siegel zeros follows from [B, Th. 5]. In the second case, the discussion in [HoL, p. 180] shows that $L(s,\mathrm{ad}^2\pi)$ factors into a Dirichlet L-function with character associated to some quadratic extension K'/K and a Hecke L-function $L_{K'}(s,\chi)$, both with conductor bounded by $\ll_K C(\pi)$. For both factors, we can bound possible Siegel zeros away from 1 by the theorem of Siegel–Brauer–Stark (see [Fo] and [St]). □

2.10 Sobolev norms. The right action of $\mathrm{GL}_2(K_\infty)$ on $L^2(\mathrm{GL}_2(K)\backslash\mathrm{GL}_2(\mathbb{A}),\omega)$ induces an action of its Lie algebra $\mathfrak{gl}(K_\infty)$ on the subspace of differentiable functions. We recall this action for the Lie subalgebra $\mathfrak{g}:=\mathfrak{sl}(K_\infty)$ generated by the independent vectors

$$H_j := \begin{pmatrix} e_j & 0 \\ 0 & -e_j \end{pmatrix}, \quad R_j := \begin{pmatrix} 0 & e_j \\ 0 & 0 \end{pmatrix}, \quad L_j := \begin{pmatrix} 0 & 0 \\ e_j & 0 \end{pmatrix}, \quad 1\leqslant j\leqslant d,$$

where $e_j=(0,\dots,0,1,0,\dots,0)$ with 1 at position j. The corresponding differential operators are (cf. [Bu, Prop. 2.2.5])

$$dH_j = -2y_j\sin(2\vartheta_j)\partial_{x_j} + 2y_j\cos(2\vartheta_j)\partial_{y_j} + \sin(2\vartheta_j)\partial_{\vartheta_j}\,, \tag{81}$$

$$dR_j = y_j\cos(2\vartheta_j)\partial_{x_j} + y_j\sin(2\vartheta_j)\partial_{y_j} + \sin^2(\vartheta_j)\partial_{\vartheta_j}\,, \tag{82}$$

$$dL_j = y_j\cos(2\vartheta_j)\partial_{x_j} + y_j\sin(2\vartheta_j)\partial_{y_j} - \cos^2(\vartheta_j)\partial_{\vartheta_j}\,. \tag{83}$$

The action of $\mathfrak{g}$ induces an action of its universal enveloping algebra $U(\mathfrak{g})$ by higher-order differential operators. This action commutes with the spectral decomposition (8), hence for each $\mathcal{D}\in U(\mathfrak{g})$ and any sufficiently smooth $\phi\in L^2(\mathrm{GL}_2(K)\backslash\mathrm{GL}_2(\mathbb{A}),\omega)$ decomposing as

$$\phi=\sum_{\pi\in\mathcal{C}_\omega}\phi_\pi+\sum_{\chi^2=\omega}\phi_\chi+\int_{\mathcal{E}_\omega}\phi_\varpi\,d\varpi$$

with $\phi_\pi\in V_\pi$, $\phi_\chi\in V_\chi$, $\phi_\varpi\in V_\varpi$, it follows that (cf. [BlH2, (33)])

$$\|\mathcal{D}\phi\|^2=\sum_{\pi\in\mathcal{C}_\omega}\|\mathcal{D}\phi_\pi\|^2+\sum_{\chi^2=\omega}\|\mathcal{D}\phi_\chi\|^2+\int_{\mathcal{E}_\omega}\|\mathcal{D}\phi_\varpi\|^2 d\varpi\,. \tag{84}$$

We now define, for any $\mu \in \mathbb{N}_0$ and any (sufficiently) smooth vector ϕ, the Sobolev norm

$$\|\phi\|_{S^\mu} := \sum_{\mathrm{ord}(\mathcal{D}) \leqslant \mu} \|\mathcal{D}\phi\|\,,$$

where $\mathcal{D}$ ranges over all monomials in $H_{j_1}, R_{j_2}, L_{j_3}$ of order at most μ in $U(\mathfrak{g})$. Clearly,

$$\|\phi\|^2_{S^\mu} \asymp_\mu \sum_{\mathrm{ord}(\mathcal{D}) \leqslant \mu} \|\mathcal{D}\phi\|^2,$$

therefore by (84) also

$$\|\phi\|^2_{S^\mu} \asymp_\mu \sum_{\pi \in \mathcal{C}_\omega} \|\phi_\pi\|^2_{S^\mu} + \sum_{\chi^2 = \omega} \|\phi_\chi\|^2_{S^\mu} + \int_{\mathcal{E}_\omega} \|\phi_\varpi\|^2_{S^\mu} d\varpi\,. \tag{85}$$

Let (π, V_π) be an automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ generated by a cusp form of arbitrary central character ω or an Eisenstein series with trivial central character $\omega = 1$, i.e. one of V_π with $\pi \in \mathcal{C}_\omega$ or $V_{\chi,\chi^{-1}}$ with χ an arbitrary Hecke character which is nontrivial on $K^{\mathrm{diag}}_{\infty,+}$. Earlier we introduced for each ideal $\mathfrak{t} \subseteq \mathfrak{o}$ an isometric embedding $R^{(\mathfrak{t})} : V_\pi(\mathfrak{c}_\pi) \hookrightarrow V_\pi$ and a vector space isomorphism $R^{(\mathfrak{t})} V_\pi(\mathfrak{c}_\pi) \to L^2(K^\times_\infty, d^\times y)$, written as $\phi \mapsto W_\phi$, which satisfy (39)–(42) and (48)–(50). Using that the right actions of $\mathrm{GL}_2(K_\infty)$ and $\mathrm{GL}_2(\mathbb{A}_{\mathrm{fin}})$ on V_π commute, it is easy to see that $U(\mathfrak{g})$ acts on each subspace $R^{(\mathfrak{t})} V_\pi(\mathfrak{c}_\pi)$ separately, which then induces an action on $L^2(K^\times_\infty, d^\times y)$. Interestingly, this action is independent of $\mathfrak{t}$. Indeed, (81)–(82) show that H_j acts by $2y_j\partial_{y_j}$ and R_j acts by $2\pi i y_j$. Then, the j-th Casimir element

$$-4\Delta_j = H_j^2 + 2R_jL_j + 2L_jR_j = H_j^2 - 2H_j + 4R_jL_j \tag{86}$$

acts by the scalar $-4\lambda_{\pi,j}$ (cf. (10) and [Bu, p. 153]), hence R_jL_j acts by $-\lambda_{\pi,j} + y_j^2\partial^2_{y_j}$ and L_j acts by $(2\pi i)^{-1}(-\lambda_{\pi,j}y_j^{-1} + y_j\partial^2_{y_j})$. These formulae justify for any $\mu \in \mathbb{N}_0$ and any 2μ times differentiable function $W : K^\times_\infty \to \mathbb{C}$ the definition of the Sobolev norm

$$\|W\|_{S^\mu} := \sum_{\mathrm{ord}(\mathcal{D}) \leqslant \mu} \|\mathcal{D}W\|\,,$$

where $\mathcal{D}$ is as before, and the bound (cf. (12) and [V2, Lem. 8.4])

$$\|W\|_{S^\mu} \ll_\mu \Big(\max_{1 \leqslant j \leqslant d} \tilde{\lambda}_{\pi,j}\Big)^\mu \|W\|_{A^{2\mu}}\,, \tag{87}$$

where

$$\|W\|_{A^\mu} := \sum_{\substack{\mu_1 + \dots + \mu_d \leqslant \mu \\ \kappa_1 \leqslant \mu_1, \dots, \kappa_d \leqslant \mu_d}} \left(\int_{K^\times_\infty} \left|\partial^{\kappa_1}_{y_1} \cdots \partial^{\kappa_d}_{y_d} W(y)\right|^2 \prod_{j=1}^d \left(|y_j| + |y_j|^{-1}\right)^{\mu_j} d^\times y\right)^{1/2}. \tag{88}$$

Lemma 4. *Let (π, V_π) be an automorphic representation of $\mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbb{A})$ as before, and let $\mathfrak{t} \subseteq \mathfrak{o}$ be an ideal. Let $a, b, c \in \mathbb{N}_0$, $0 < \varepsilon < 1/4$, and θ as in (11). Let $P \in \mathbb{C}[x_1, \dots, x_d]$ be a polynomial of degree at most a in each variable, and consider the differential operator $\mathcal{D} := P(y_1\partial_{y_1}, \dots, y_d\partial_{y_d})$. Then for $\phi \in R^{(\mathfrak{t})} V_\pi(\mathfrak{c}_\pi)$*

and $y \in K_\infty^\times$ *we have, using the notation* (12),

$$\mathcal{D}W_\phi(y) \ll_{a,b,c,P,K,\varepsilon} (\mathcal{N}\mathfrak{t})^\varepsilon (\mathcal{N}\mathfrak{c}_\pi)^\varepsilon (\mathcal{N}\tilde{\lambda}_\pi)^{-c} \|\phi\|_{S^{d(5+a+b+2c)}} \prod_{j=1}^d |y_j|^{1/2-\theta-\varepsilon} \min\left(1, |y_j|^{-b}\right).$$

Proof. To start with, let us fix $q \in \mathbb{Z}^d$ and assume $\phi \in R^{(\mathfrak{t})}V_{\pi,q}(\mathfrak{c}_\pi)$. By (25), (29), (35), (41), (69) we have

$$|W_\phi(y)| = \|W_\phi\| \left|\tilde{W}_{q/2,\nu_\pi}(y)\right|, \quad y \in K_\infty^\times, \tag{89}$$

where by (42), (50), (13),

$$\|W_\phi\| \ll_{K,\varepsilon} (\mathcal{N}\mathfrak{t})^\varepsilon (\mathcal{N}\mathfrak{c}_\pi)^\varepsilon (\mathcal{N}\tilde{\nu}_\pi)^\varepsilon \|\phi\|.$$

In addition, (82)–(83) show that $(R_j - L_j)\phi = iq\phi$. Now we infer, using (27)–(29), that

$$W_\phi(y) \ll_{K,\varepsilon} (\mathcal{N}\mathfrak{t})^\varepsilon (\mathcal{N}\mathfrak{c}_\pi)^\varepsilon (\mathcal{N}\tilde{\nu}_\pi)^\varepsilon \|\phi\| \prod_{j=1}^d \left(\tilde{\nu}_{\pi,j} + |q_j|\right)^{1+\theta} \left(|y_j|^{1/2-\varepsilon} + |y_j|^{1/2-\theta-\varepsilon}\right)$$

$$\ll_{K,\varepsilon} (\mathcal{N}\mathfrak{t})^\varepsilon (\mathcal{N}\mathfrak{c}_\pi)^\varepsilon (\mathcal{N}\tilde{\nu}_\pi)^{1+\theta+\varepsilon} \left\| \prod_{j=1}^d \left(1 - (R_j - L_j)^2\right)\phi \right\| \prod_{j=1}^d \frac{|y_j|^{1/2-\varepsilon} + |y_j|^{1/2-\theta-\varepsilon}}{(1+|q_j|)^{1-\theta}}.$$

For an arbitrary $\phi \in R^{(\mathfrak{t})}V_\pi(\mathfrak{c}_\pi)$ with weight decomposition (cf. (16)–(17))

$$\phi = \sum_{q \in \mathbb{Z}^d} \phi_q, \quad \phi_q \in R^{(\mathfrak{t})}V_{\pi,q}(\mathfrak{c}_\pi),$$

we apply the operator $\mathcal{D}' := \prod_{j=1}^d \left(1 - (R_j - L_j)^2\right)$ on both sides obtaining the weight decomposition

$$\mathcal{D}'\phi = \sum_{q \in \mathbb{Z}^d} \mathcal{D}'\phi_q, \quad \mathcal{D}'\phi_q = (1+q^2)\phi_q \in R^{(\mathfrak{t})}V_{\pi,q}(\mathfrak{c}_\pi).$$

In particular, $\|\mathcal{D}'\phi\|^2 = \sum_{q \in \mathbb{Z}^d} \|\mathcal{D}'\phi_q\|^2$, hence the previous bound and Cauchy–Schwarz yield

$$W_\phi(y) \ll_{K,\varepsilon} (\mathcal{N}\mathfrak{t})^\varepsilon (\mathcal{N}\mathfrak{c}_\pi)^\varepsilon (\mathcal{N}\tilde{\lambda}_\pi) \|\phi\|_{S^{2d}} \prod_{j=1}^d \left(|y_j|^{1/2-\varepsilon} + |y_j|^{1/2-\theta-\varepsilon}\right).$$

Depending on $a, b, c \in \mathbb{N}_0$ and $y \in K_\infty^\times$, we replace ϕ by $\mathcal{D}''\phi \in R^{(\mathfrak{t})}V_\pi(\mathfrak{c}_\pi)$ with

$$\mathcal{D}'' := \Big(\prod_{1 \leqslant j \leqslant d} (1/2 + H_j^2 + 2R_jL_j + 2L_jR_j)^{c+1} \Big) \Big(\prod_{\substack{1 \leqslant j \leqslant d \\ |y_j| > 1}} R_j^{b+1} \Big) P(H_1, \ldots, H_d),$$

then we obtain the general bound of the lemma by combining (86) and $|1/2 - 4\lambda_{\pi,j}| > \tilde{\lambda}_{\pi,j}/5$ for each $1 \leqslant j \leqslant d$. □

Lemma 5. *Let* (π, V_π) *be an irreducible cuspidal representation of* $\mathrm{GL}_2(K)\backslash\mathrm{GL}_2(\mathbb{A})$ *with unitary central character, and let* $\phi \in V_\pi(\mathfrak{c}_\pi)$ *be such that* $\|\phi\|_{S^{3d}}$ *exists. Then*

$$\|\phi\|_\infty := \sup_{g \in \mathrm{GL}_2(\mathbb{A})} |\phi(g)| \ll_{\pi,K} \|\phi\|_{S^{3d}}.$$

Remark 8. It is relatively easy to show that the implied constant depends polynomially on $C(\pi)$, and the order of the Sobolev norm could also be lowered easily. However, it seems hard and would be interesting to find close to optimal bounds for the sup-norm of an automorphic form in terms of the L^2-norm (or some small Sobolev norm) and the various parameters of π. For strong results in this direction see the work of Bernstein and Reznikov [BeR].

Proof. Let us first assume that $\phi \in V_{\pi,q}(\mathfrak{c}_\pi)$, i.e. ϕ is of pure weight $q \in \mathbb{Z}^d$. Let $g \in \mathrm{GL}_2(\mathbb{A})$, and let $i_1, \dots, i_h \in \mathbb{A}_{\mathrm{fin}}^\times$ be h finite ideles representing the ideal classes of K. By strong approximation [Bu, Th. 3.3.1] there are $\gamma \in \mathrm{GL}_2(K)$, $g' \in \mathrm{GL}_2(K_\infty)$, and $k \in \mathcal{K}(\mathfrak{o})$ such that

$$g = \gamma \left(g' \times \begin{pmatrix} i_j & 0 \\ 0 & 1 \end{pmatrix} k \right)$$

for some $1 \leqslant j \leqslant h$. It follows from [Fr, p. 36 & p. 67] that there are elements $a_1, \dots, a_{2^d h} \in \mathrm{GL}_2(K)$ regarded as elements of $\mathrm{GL}_2(K_\infty)$ and some $\delta > 0$ depending only on K such that, for suitable $z \in Z(K_\infty)$, $\gamma' \in \mathrm{SL}_2(\mathfrak{o})$ regarded as an element of $\mathrm{SL}_2(K_\infty)$, and $k' \in \mathrm{SO}_2(K_\infty)$, we have

$$g' = z\gamma' a_{j'} \begin{pmatrix} y' & x' \\ 0 & 1 \end{pmatrix} k'$$

for some $1 \leqslant j' \leqslant 2^d h$ and some $\left(\begin{smallmatrix} y' & x' \\ 0 & 1 \end{smallmatrix}\right) \in P(K_\infty)$ with $y'_1, \dots, y'_d > \delta$. Combining with the previous display we obtain

$$g = z\gamma\gamma' a_{j'} \left(\begin{pmatrix} y' & x' \\ 0 & 1 \end{pmatrix} k' \times a_{j'}^{-1} \gamma'^{-1} \begin{pmatrix} i_j & 0 \\ 0 & 1 \end{pmatrix} k \right),$$

where $z \in Z(K_\infty)$ is now regarded as an element of $Z(\mathbb{A})$, and the first (resp. second) occurrences of γ' and $a_{j'}$ are regarded as elements of $\mathrm{GL}_2(\mathbb{A})$ (resp. of $\mathrm{GL}_2(\mathbb{A}_{\mathrm{fin}})$). Here $\gamma\gamma' a_{j'} \in \mathrm{GL}_2(K)$, while $a_{j'}^{-1}\gamma'^{-1} \left(\begin{smallmatrix} i_j & 0 \\ 0 & 1 \end{smallmatrix}\right) k$ lies in a fixed compact subset of $\mathrm{GL}_2(\mathbb{A}_{\mathrm{fin}})$ depending only on K which can be covered by finitely many left cosets of the open subgroup $\mathcal{K}(\mathfrak{c}_\pi)$. It follows that

$$g = z\tilde{\gamma} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} (\tilde{k}_\infty \times \tilde{k}_{\mathrm{fin}})$$

for some $\tilde{\gamma} \in \mathrm{GL}_2(K)$, $\tilde{k} = \tilde{k}_\infty \times \tilde{k}_{\mathrm{fin}} \in \mathrm{SO}_2(K_\infty) \times \mathcal{K}(\mathfrak{c}_\pi)$, and $\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \in P(\mathbb{A})$, where $y = y_\infty \times y_{\mathrm{fin}}$ is such that all coordinates of y_∞ exceed δ and y_{fin} takes values from a finite set depending only on K and $\mathfrak{c}_\pi$. Thus the Fourier expansion (34) together with (33), (89) and Lemma 3 gives

$$\begin{aligned} |\phi(g)| = \left| \phi\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right) \right| &\leqslant \sum_{\substack{r \in (y_{\mathrm{fin}}^{-1}) \\ r \neq 0}} \frac{|\lambda_\pi(r y_{\mathrm{fin}})|}{\sqrt{\mathcal{N}(r y_{\mathrm{fin}})}} \left| W_\phi(r y_\infty) \right| \\ &\ll_{\pi,K} \|\phi\| \sum_{\substack{r \in (y_{\mathrm{fin}}^{-1}) \\ r \neq 0}} \left| \tilde{W}_{q/2,\nu_\pi}(r y_\infty) \right|, \end{aligned}$$

where $(y_{\rm fin}^{-1}) = y_{\rm fin}^{-1}\mathfrak{o}$ is the fractional ideal corresponding to $y_{\rm fin}^{-1}$. We fix some $0 < \varepsilon < 1/20$ and let

$$R_1 := \left\{ r \in (y_{\rm fin}^{-1}) \;\middle|\; |r^{\sigma_j}| < \delta^{-1}\big(|q_j| + |\nu_{\pi,j}| + 1\big) \prod_{j=1}^{d} \big(|r^{\sigma_j}| + |q_j|\big)^{\varepsilon} \right\},$$

$$R_2 := (y_{\rm fin}^{-1}) - R_1\,.$$

Using (26)–(29) and the property of y_∞, we find

$$\sum_{\substack{r\in R_2\\ r\neq 0}} \big|\tilde{W}_{q/2,\nu_\pi}(ry_\infty)\big| \ll_{\pi,K,\varepsilon} 1$$

and

$$\sum_{\substack{r\in R_1\\ r\neq 0}} \big|\tilde{W}_{q/2,\nu_\pi}(ry_\infty)\big| \ll_{\pi,K} \#R_1 \prod_{j=1}^{d} \big(1+|q_j|\big)^{1+\theta} \ll_{\pi,K,\varepsilon} \prod_{j=1}^{d} \big(1+|q_j|\big)^{2+\theta+\varepsilon}.$$

By (82)–(83) we see now that for any $\phi \in V_{\pi,q}(\mathfrak{c}_\pi)$ we have

$$\|\phi\|_\infty \ll_{\pi,K} \left\| \prod_{j=1}^{d} (1+R_j-L_j)^3 \phi \right\| \prod_{j=1}^{d} \big(1+|q_j|\big)^{-2/3}.$$

Using Cauchy–Schwarz and Parseval we can infer for a general $\phi \in V_\pi(\mathfrak{c}_\pi)$ that

$$\|\phi\|_\infty \ll_{\pi,K} \|\phi\|_{S^{3d}}\,,$$

assuming the right-hand side exists. □

2.11 Waldspurger's theorem and generalizations. Let $r \in \mathfrak{o}$ be a nonzero squarefree integer, i.e. $0 \leqslant v_{\mathfrak{p}}(r) \leqslant 1$ for all prime ideals $\mathfrak{p} \subseteq \mathfrak{o}$. If χ_r denotes the quadratic character associated to the extension $K(\sqrt{r})/K$, the central value $L(1/2, \pi \otimes \chi_r)$ is related to the square of the r-th Fourier coefficient of a half-integral weight Hilbert modular form. The prototype of such a theorem for $K = \mathbb{Q}$ goes back to Waldspurger [W2] with refinements by Kohnen–Zagier [KoZ], [Ko]. For an arbitrary totally real number field K, precise results of this type can be found for example in [K, Th. 8.1] and [BaM, Th. 4.3]. Using these, one can turn a bound for twisted central L-values into a bound for the Fourier coefficients of a half-integral weight Hilbert modular form. An explicit statement of this phenomenon is [BaM, Th. 1.5] which we recall below.

Let $\widetilde{\rm SL}_2$ denote the metaplectic double cover of ${\rm SL}_2$, and let $(\tilde\pi, V_{\tilde\pi})$ be an irreducible cuspidal representation of $\widetilde{\rm SL}_2(K)\backslash\widetilde{\rm SL}_2(\mathbb{A})$ orthogonal to the theta series generated by quadratic forms in one variable. Let π be the unique irreducible cuspidal representation of ${\rm GL}_2(K)Z(\mathbb{A})\backslash {\rm GL}_2(\mathbb{A})$ associated to $\tilde\pi$ by the Shimura–Waldspurger correspondence [W1,3]. Define the r-th Fourier coefficient of a smooth vector $\tilde\phi \in V_{\tilde\pi}$ as

$$\tilde{W}^r_{\tilde\phi} := \int_{K\backslash\mathbb{A}} \tilde\phi\left(\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\right) \psi(-rx)dx\,.$$

Assume that $\tilde\phi$ is a pure tensor $\otimes_v \tilde\phi_v$, and for each archimedean place $1 \leqslant j \leqslant d$ define a quantity $e(\tilde\phi_j, r)$ as in [BaM, (4.3)], cf. also [BaM, §2.2]. For $\tilde\phi_j$ belonging to the holomorphic discrete series, this quantity is calculated explicitly in [BaM, Prop. 8.8]. Assume that there is a bound

$$L(1/2,\pi\otimes\chi_r)\ll_{\pi,K}|\mathcal{N}r|^{\beta}$$

for some $\beta>0$, then one has

$$\tilde{W}_{\tilde{\phi}}^{r}\prod_{j=1}^{d}e(\tilde{\phi}_j,r)\ll_{\tilde{\phi},K}|\mathcal{N}r|^{\frac{\beta-1}{2}}. \tag{90}$$

By the last two displays, Corollary 1 is an immediate consequence of Theorem 1.

2.12 Kuznetsov's formula. There are several adelic [Y], [KniL2] and classical [BrMP] versions of the Kuznetsov formula over number fields available in the literature. For our purposes, a slightly generalized version of the "semi-classical" formula given in [V1] (which in turn is based on [BrM]) is the most suitable. The extension is needed because [V1] deals only with representations that are spherical at infinity (i.e. totally even non-exceptional Hilbert–Maaß forms), while we need to include holomorphic forms and totally even exceptional Hilbert–Maaß forms. Fortunately, the necessary integral transforms together with sharp estimates are provided in full detail in [BrMP], so we can quote the results and restrict ourselves to a brief exposition.

We introduce the set

$$\mathcal{S}:=\left\{\nu\in\mathbb{C}:|\Re\nu|<\tfrac{2}{3}\right\}\cup\left(\tfrac{1}{2}+\mathbb{Z}\right),$$

and for each $1\leqslant j\leqslant d$ we consider an even function $k_j:\mathcal{S}\to\mathbb{C}$, holomorphic on the interior of $\mathcal{S}$, which satisfies the decay condition $k_j(\nu)\ll(1+|\nu|)^{-a}$ for some $a>2$. We write

$$k(\nu):=\prod_j k_j(\nu_j),\quad \nu\in\mathcal{S}^d.$$

Following [BrMP, Defs. 2.5.2-2.5.4 & (25)], we define the Bessel transforms

$$\check{k}_j(t):=-i\int_{(0)}k_j(\nu)J_{2\nu}\big(4\pi\sqrt{t}\big)\frac{\nu\,d\nu}{\cos(\pi\nu)} + \sum_{b\geqslant 2\text{ even}}(-1)^{b/2}(b-1)k_j\left(\frac{b-1}{2}\right)J_{b-1}\big(4\pi\sqrt{t}\big),\quad t>0;$$

$$\check{k}_j(t):=-i\int_{(0)}k_j(\nu)I_{2\nu}\big(4\pi\sqrt{|t|}\big)\frac{\nu\,d\nu}{\cos(\pi\nu)},\quad t<0;$$

$$\tilde{k}_j:=\frac{i}{2}\int_{(0)}k_j(\nu)\nu\tan(\pi\nu)d\nu+\sum_{b\geqslant 2\text{ even}}\frac{b-1}{2}k_j\left(\frac{b-1}{2}\right).$$

Let $\mathfrak{c},\mathfrak{y}_1,\mathfrak{y}_2\subseteq\mathfrak{o}$ be nonzero ideals. Within a fixed set of representatives of all ideal classes of K we define C as the subset of ideals $\mathfrak{a}$ satisfying $\mathfrak{a}^2\mathfrak{y}_1\mathfrak{y}_2\mathfrak{d}^2\sim 1$. For each $\mathfrak{a}\in C$ we fix, once and for all, a generator γ of the principal ideal $\mathfrak{a}^2\mathfrak{y}_1\mathfrak{y}_2\mathfrak{d}^2$. For any nonzero elements $c\in\mathfrak{c}\mathfrak{a}^{-1}$, $r_1\in\mathfrak{y}_1$, $r_2\in\mathfrak{y}_2$ we define the Kloosterman sum

$$S(r_1,\mathfrak{y}_1;r_2,\mathfrak{y}_2;c,\mathfrak{a}):=\mathrm{KS}\big(r_1,(\mathfrak{y}_1\mathfrak{d})^{-1};r_2\gamma^{-1},(\mathfrak{y}_2\mathfrak{d})^{-1};c,\mathfrak{a}\big),$$

where the right-hand side is given by [V1, Def. 2]. We only need to know Weil's bound for this type of Kloosterman sum [V1, (13)]

$$S(r_1,\mathfrak{y}_1;r_2,\mathfrak{y}_2;c,\mathfrak{a}) \ll_{K,\varepsilon} \big(\mathcal{N}\gcd(r_1\mathfrak{y}_1^{-1},r_2\mathfrak{y}_2^{-1},c\mathfrak{a})\big)^{1/2}(\mathcal{N}(c\mathfrak{a}))^{1/2+\varepsilon}. \qquad (91)$$

Since we will not need the details later, we suppress a detailed discussion of the continuous spectrum contribution and follow [V1] to abbreviate this quantity (whose exact shape is irrelevant for our purposes) by CSC. Then we have the following variant of Kuznetsov's formula:

$$\begin{aligned}
&\big[\mathcal{K}(\mathfrak{o}):\mathcal{K}(\mathfrak{c})\big]^{-1}\sum_{\substack{\pi\in\mathcal{C}(\mathfrak{c})\\ \varepsilon_\pi=1}} C_\pi^{-1}\sum_{\mathfrak{t}\mid\mathfrak{c}\mathfrak{c}_\pi^{-1}} k(\nu_\pi)\bar\lambda_\pi^{(\mathfrak{t})}(r_1\mathfrak{y}_1^{-1})\lambda_\pi^{(\mathfrak{t})}(r_2\mathfrak{y}_2^{-1}) + \mathrm{CSC}\\
&\quad= c_1\delta(r_1\mathfrak{y}_1^{-1},r_2\mathfrak{y}_2^{-1})\prod_{j=1}^d \tilde{k}_j\\
&\qquad+c_2\sum_{\mathfrak{a}\in C}\sum_{u\in U/U^2}\sum_{c\in\mathfrak{c}\mathfrak{a}^{-1}}\frac{S(r_1,\mathfrak{y}_1;ur_2,\mathfrak{y}_2;c,\mathfrak{a})}{\mathcal{N}(c\mathfrak{a})}\prod_{j=1}^d \check{k}_j\left(\left(\frac{ur_1r_2}{\gamma c^2}\right)^{\sigma_j}\right). \qquad (92)
\end{aligned}$$

Here π runs over all totally even cuspidal representations of trivial central character and conductor dividing $\mathfrak{c}$ (cf. (22) and (30)); C_π was defined in (37) and estimated in Lemma 3; the coefficients $\lambda_\pi^{(\mathfrak{t})}$ are those in (40); c_1, c_2 are certain positive constants depending *only* on K; finally $\delta(\mathfrak{a},\mathfrak{b})=1$ if and only if $\mathfrak{a}=\mathfrak{b}$.

We will only discuss the main ideas of the proof, since all ingredients can be found in detail in [V1], [BrM], [BrMP]. The transition between the classical and adelic versions of the Kuznetsov formula is based on the fact that for any nonzero ideal $\mathfrak{y}\subseteq\mathfrak{o}$ there is an embedding of coset spaces

$$\Gamma(\mathfrak{y},\mathfrak{c})Z(K_\infty)\backslash\operatorname{GL}_2(K_\infty)\hookrightarrow \operatorname{GL}_2(K)Z(K_\infty)\backslash\operatorname{GL}_2(\mathbb{A})/\mathcal{K}(c)$$

given by

$$\Gamma(\mathfrak{y},\mathfrak{c})Z(K_\infty)g\mapsto \operatorname{GL}_2(K)Z(K_\infty)g\begin{pmatrix}\eta^{-1}&0\\0&1\end{pmatrix}\mathcal{K}(\mathfrak{c}),\quad g\in\operatorname{GL}_2(K_\infty),$$

where $\eta\in\mathbb{A}_{\mathrm{fin}}^\times$ is any finite idele representing $\mathfrak{y}$. Indeed, it is straightforward to see that the above map is well-defined and injective by combining (2) with

$$\prod_{\mathfrak{p}}\mathcal{K}(\mathfrak{y}_\mathfrak{p},\mathfrak{c}_\mathfrak{p})=\begin{pmatrix}\eta^{-1}&0\\0&1\end{pmatrix}\mathcal{K}(\mathfrak{c})\begin{pmatrix}\eta&0\\0&1\end{pmatrix}.$$

In addition, strong approximation [Bu, Th. 3.3.1] shows that the image consists of the double cosets whose union equals (cf. section 2.1.1)

$$\big\{g\in\operatorname{GL}_2(\mathbb{A})\mid \det(g)\in\eta^{-1}K^\times K_\infty^\times\Omega\big\}.$$

Therefore if $\mathfrak{y}_1,\dots,\mathfrak{y}_h$ represent the ideal classes of K, then we obtain a decomposition of spaces (cf. [V1, §6.1])

$$\operatorname{GL}_2(K)Z(K_\infty)\backslash\operatorname{GL}_2(\mathbb{A})/\mathcal{K}(c)\cong\coprod_{j=1}^h\Gamma(\mathfrak{y}_j,\mathfrak{c})Z(K_\infty)\backslash\operatorname{GL}_2(K_\infty).$$

The Haar measure on $Z(K_\infty)\backslash\operatorname{GL}_2(\mathbb{A})$ defined in section 2.1.3 gives rise to a Borel measure on the left-hand side assigning to each Borel set the measure of its preimage in $\mathrm{GL}_2(K)Z(K_\infty)\backslash\mathrm{GL}_2(\mathbb{A})$ under the natural projection. The Haar measure on $Z(K_\infty)\backslash\mathrm{GL}_2(K_\infty)$ defined in section 2.1.3 induces a Borel measure on the right-hand side. Now an important feature is that the measure of each Borel set on the left-hand side is exactly $[\mathcal{K}(\mathfrak{o}):\mathcal{K}(\mathfrak{c})]^{-1}$ times the measure of the corresponding Borel set on the right-hand side. To see this claim, it suffices to show that for any nonzero ideal $\mathfrak{y}\subseteq\mathfrak{o}$ and for any Borel set $U\subseteq\mathrm{GL}_2(K_\infty)$ representing distinct cosets $\Gamma(\mathfrak{y},\mathfrak{c})Z(K_\infty)g$ for $g\in U$, we have

$$\operatorname{vol}\left(\mathrm{GL}_2(K)Z(K_\infty)\backslash\mathrm{GL}_2(K)Z(K_\infty)U\begin{pmatrix}\eta^{-1}&0\\0&1\end{pmatrix}\mathcal{K}(\mathfrak{c})\right)\\ =\left[\mathcal{K}(\mathfrak{o}):\mathcal{K}(\mathfrak{c})\right]^{-1}\operatorname{vol}\left(Z(K_\infty)\backslash Z(K_\infty)U\right).$$

By the discussion above, the double cosets $\mathrm{GL}_2(K)Z(K_\infty)g\left(\begin{smallmatrix}\eta^{-1}&0\\0&1\end{smallmatrix}\right)\mathcal{K}(\mathfrak{c})$ for $g\in U$ are distinct, hence the left-hand side equals

$$\operatorname{vol}\left(Z(K_\infty)\backslash Z(K_\infty)U\begin{pmatrix}\eta^{-1}&0\\0&1\end{pmatrix}\mathcal{K}(\mathfrak{c})\right)\\ =\operatorname{vol}(Z(K_\infty)\backslash Z(K_\infty)U)\operatorname{vol}\left(\begin{pmatrix}\eta^{-1}&0\\0&1\end{pmatrix}\mathcal{K}(\mathfrak{c})\right)\\ =\operatorname{vol}\bigl(Z(K_\infty)\backslash Z(K_\infty)U\bigr)\operatorname{vol}(\mathcal{K}(\mathfrak{c}))\,.$$

Since $\operatorname{vol}(K(\mathfrak{c}))=[\mathcal{K}(\mathfrak{o}):\mathcal{K}(\mathfrak{c})]^{-1}$ by $\operatorname{vol}(K(\mathfrak{o}))=1$, our claim follows.

Let T denote the subgroup of elements $\left(\begin{smallmatrix}*&0\\0&1\end{smallmatrix}\right)\in\mathrm{O}_2(K_\infty)$, i.e. those with coordinates $\left(\begin{smallmatrix}\pm1&0\\0&1\end{smallmatrix}\right)$. Then T represents $\mathrm{O}_2(K_\infty)/\mathrm{SO}_2(K_\infty)$ and the above discussion shows

$$\mathrm{GL}_2(K)Z(K_\infty)\backslash\mathrm{GL}_2(\mathbb{A})/T\mathcal{K}(c)\cong\coprod_{j=1}^{h}\Gamma(\mathfrak{y}_j,\mathfrak{c})Z(K_\infty)\backslash\mathrm{GL}_2(K_\infty)/T$$

with similarly related Borel measures on the two sides. We denote by FS the L^2-space of the left-hand side, viewed as a Hilbert space of measurable functions $\phi:\mathrm{GL}_2(\mathbb{A})\to\mathbb{C}$ which are left $\mathrm{GL}_2(K)Z(K_\infty)$-invariant and right $T\mathcal{K}(\mathfrak{c})$-invariant. This space is analogous to FS in [V1, §2.3] for the special case $\chi=1$, the only difference being that instead of right $\mathrm{O}_2(K_\infty)$-invariance we require right T-invariance. We clearly have

$$\mathrm{FS}\cong\bigoplus_{j=1}^{h}L^2\bigl(\Gamma(\mathfrak{y}_j,\mathfrak{c})Z(K_\infty)\backslash\mathrm{GL}_2(K_\infty)/T\bigr)\,,\tag{93}$$

and in order to derive (92), we follow [V1, §6]. The proof is based on a geometric and spectral evaluation of a certain inner product formed of two Poincaré series on FS, each of which is supported in only one component on the right-hand side of (93). The spectral expansion is carried out in an orthonormal basis of the right-hand side of (93) which, according to our discussion above, is provided by $[\mathcal{K}(\mathfrak{o}):\mathcal{K}(\mathfrak{c})]^{-1/2}$ times any orthonormal basis of FS. For the latter we make use of the decomposition

$$\mathrm{FS}=\bigoplus_{\omega\in\widehat{C(K)}}L^2\bigl(\mathrm{GL}_2(K)\backslash\mathrm{GL}_2(\mathbb{A})/T\mathcal{K}(\mathfrak{c}),\omega\bigr)\,,\tag{94}$$

where each class group character ω is regarded as a character of $\mathbb{A}^\times$ trivial on $K^\times K_\infty^\times\Omega$, and the corresponding component is the right $T\mathcal{K}(\mathfrak{c})$-invariant subspace of $L^2(\mathrm{GL}_2(K)\backslash\mathrm{GL}_2(\mathbb{A}),\omega)$, in the notation of section 2.2. Utilizing (8), (39), (47), we can form an orthonormal basis of FS from certain totally even automorphic forms of pure weight, level $\mathfrak{c}$ and central character trivial on $K^\times K_\infty^\times\Omega$. By averaging over $C(K)$ several inner products associated with the same pair of Poincaré series, we can ensure that only $\omega=1$ contributes to the final spectral expansion. This outline explains the structure of the left-hand side of (92).

To carry out the above plan, we need to work with slightly more general Poincaré series than [V1, (89)], namely we only require right T-invariance instead of right $\mathrm{O}_2(K_\infty)$-invariance. Then the geometric evaluation [V1, §6.3] goes through with no changes, but in the spectral evaluation [V1, §6.4] the integrals over A have to be replaced by integrals over $A\mathrm{O}_2(K_\infty)$. Now we choose the test function f as in [BrMP, §5.3, see also Def. 5.2.4]. The special integrals in [V1, (96a), (96b), (98)] are evaluated in [V1, §6.5] using the relations [BrM, (25), Prop. 9.4, (26)], respectively. In our more general situation, we use the corresponding results [BrMP, (83), Prop. 5.2.6, (84)]. This yields the formula (92) as in [V1, §6.6]. We note that in [BrMP] the authors are faced with more subtle convergence issues; this is, for example, reflected in the fact that in [BrMP, Def. 5.2.4] weight 2 Maaß forms rather than weight 0 Maaß forms are used which leads to the correction factor $\left(\frac{1}{4}-\nu^2\right)^{-1}$ in [BrMP, (87)].

Remark 9. Unfortunately, there are different concurrent normalizations in the literature which makes it a little tedious to compare the various papers. For the convenience of the reader we give an account of the differences. There are three sources of different notation/normalization:

- *Groups.* As mentioned in section 2.1.2, our congruence subgroups are slightly different from those in [V1]; our $\mathcal{K}(\mathfrak{y},\mathfrak{c})\subseteq\mathrm{GL}_2(K_\mathfrak{p})$ is precisely the group $K_{0,\mathfrak{p}}(\mathfrak{c},(\mathfrak{y}\mathfrak{d})^{-1})$ defined in [V1, §2.2].
- *Measures.* In [BrM], [BrMP] the group $N(K_\infty)$ of upper triangular unipotent matrices is equipped with the measure $\pi^{-d}dx_1\cdots dx_d$ (with dx the usual Lebesgue measure) whereas we have normalized the measure in section 2.1.3 as $|D_K|^{-1/2}dx_1\cdots dx_d$. Venkatesh [V1] follows the normalization in [BrM], [BrMP]. (We remark, however, that a comparison of [V1, (11)] and [BrM, (5)] shows that the factor $\mathrm{vol}(\Gamma_N\backslash N)^{-1}$ in [BrM, (5)] is wrongly adapted in [V1, (11)] as $(\mathcal{N}\mathfrak{d})^{-1/2}$ instead of $2^{d_\mathbb{C}}\pi^{d_\mathbb{R}+d_\mathbb{C}}(\mathcal{N}\mathfrak{d})^{-1/2}$, cf. [V1, (96)].)
- *Whittaker functions.* Our normalization of Whittaker functions coincides with that of [BrMP] for $\Re\nu=0$, up to a factor of absolute value $\sqrt{2\pi}$ at each archimedean place, cf. (24) and [BrMP, (16)]. If $\Re\nu\neq 0$, the discrepancy between [BrMP, (16)] and our definition (23) is compensated by [BrMP, (15)]. In [BrM], [V1] only the weight 0 case is treated, and hence the authors use $\sqrt{y}K_\nu(2\pi y)=\frac{1}{2}W_{0,\nu}(4\pi y)$ (unnormalized!) as a Whittaker function. This scales the Fourier coefficients up by a factor $\pi^{-1}(2\cos(\pi\nu))^{1/2}$, cf. also the remark before [BrMP, Def. 2.5.2]. Accordingly, the decay conditions of the test function in [BrMP] do not include exponentials, and the measure in [BrMP, Def. 2.5.2] contains the function $\tan(\pi\nu)$ rather than $\sin(\pi\nu)$.

As a first application of the Kuznetsov formula and a warm-up for later calculations we will deduce a weak Weyl law that will give an upper bound of roughly the expected order of magnitude.

LEMMA 6. *Let $\mathfrak{c},\mathfrak{m}\subseteq\mathfrak{o}$ be two nonzero ideals, let $X\in[1,\infty)^d$, and write $\Xi:=\prod_{j=1}^d X_j$. Then for any $\varepsilon>0$ we have*

$$\sum_{\substack{\pi\in\mathcal{C}(\mathfrak{c})\\ \varepsilon_\pi=1\\ |\nu_{\pi,j}|\leqslant X_j}} 1 \ll_{K,\varepsilon} \Xi^{2+\varepsilon}(\mathcal{N}\mathfrak{c})^{1+\varepsilon}$$

and

$$\sum_{\substack{\pi\in\mathcal{C}(\mathfrak{c})\\ \varepsilon_\pi=1\\ |\nu_{\pi,j}|\leqslant X_j}} \sum_{\mathfrak{t}\mid\mathfrak{c}\mathfrak{c}_\pi^{-1}} \left|\lambda_\pi^{(\mathfrak{t})}(\mathfrak{m})\right|^2 \ll_{K,\varepsilon} \Xi^{2+\varepsilon}\big((\mathcal{N}\mathfrak{c})^{1+\varepsilon}+(\mathcal{N}\gcd(\mathfrak{m},\mathfrak{c}))^{1/2}(\mathcal{N}\mathfrak{m})^{1/2+\varepsilon}\big)\,.$$

REMARK 10. This should be compared with Lemma 2. The first bound with unspecified exponents is contained in [MV, (9.3)].

Proof. The first bound follows from the second with $\mathfrak{m}=\mathfrak{o}$ by noting that $\lambda_\pi^{(\mathfrak{o})}(\mathfrak{o})=\lambda_\pi(\mathfrak{o})=1$. To prove the second bound, we choose an ideal class representative $\mathfrak{y}\sim\mathfrak{m}^{-1}$ from a fixed set depending on K, then $\mathfrak{m}=r\mathfrak{y}^{-1}$ with $r\in\mathfrak{y}$ and $\mathcal{N}(r)\asymp_K\mathcal{N}\mathfrak{m}$. We apply Kuznetsov's formula (92) with $r_1=r_2=r$, $\mathfrak{y}_1=\mathfrak{y}_2=\mathfrak{y}$, $k(\nu):=\prod_{j=1}^d k_{X_j}(\nu_j)$, where for any $Z\geqslant 1$

$$k_Z(\nu):=\begin{cases} e^{(\nu^2-1/4)/Z^2}, & |\Re\nu|<\frac{2}{3}\,,\\ 1\,, & \nu\in\frac{1}{2}+\mathbb{Z}\text{ and }\frac{3}{2}\leqslant|\nu|\leqslant Z\,. \end{cases} \tag{95}$$

By [BrMP, p. 124–126] we have

$$\check{k}(t)\ll Z^2\min\big(1,|t|^{1/2}\big)\quad\text{and}\quad\tilde{k}\ll Z^2. \tag{96}$$

By Lemma 3 we have $[\mathcal{K}(\mathfrak{o}):\mathcal{K}(\mathfrak{c})]C_\pi\ll_{K,\varepsilon}\Xi^\varepsilon(\mathcal{N}\mathfrak{c})^{1+\varepsilon}$ for the relevant π. Hence the diagonal term contributes $\ll_{K,\varepsilon}\Xi^{2+\varepsilon}(\mathcal{N}\mathfrak{c})^{1+\varepsilon}$. By (91) and (96), the off-diagonal contribution is at most

$$\ll_{K,\varepsilon}\Xi^{2+\varepsilon}(\mathcal{N}\mathfrak{c})^{1+\varepsilon}\max_{\mathfrak{a}\in C}\sum_{0\neq c\in\mathfrak{c}\mathfrak{a}^{-1}}\frac{(\mathcal{N}\gcd(\mathfrak{m},c\mathfrak{a}))^{1/2}}{(\mathcal{N}(c))^{1/2-\varepsilon}}\prod_{j=1}^d\min\left(1,\left|\left(\frac{r^2}{c^2}\right)^{\sigma_j}\right|^{1/2}\right).$$

We now use [BrM, Lem. 8.1] as in the proof of [BrMP, Lem. 3.2.1]. We infer that the c-sum is

$$\ll_K\sum_{0\neq(c)\subseteq\mathfrak{c}\mathfrak{a}^{-1}}\frac{(\mathcal{N}\gcd(\mathfrak{m},c\mathfrak{a}))^{1/2}}{(\mathcal{N}(c))^{1/2-\varepsilon}}\left(1+\left|\log\frac{\mathcal{N}(c^2)}{\mathcal{N}(r^2)}\right|^{d-1}\right)\min\left(1,\left(\frac{\mathcal{N}(r^2)}{\mathcal{N}(c^2)}\right)^{1/2}\right)$$

$$\ll_{K,\varepsilon}\sum_{0\neq(c)\subseteq\mathfrak{c}\mathfrak{a}^{-1}}\frac{(\mathcal{N}\gcd(\mathfrak{m},c\mathfrak{a}))^{1/2}}{(\mathcal{N}(c))^{1/2-\varepsilon}}\left(\frac{\mathcal{N}(r^2)}{\mathcal{N}(c^2)}\right)^{1/4+\varepsilon}.$$

The last sum extends in a natural fashion to all nonzero ideals contained in $\mathfrak{c}\mathfrak{a}^{-1}$, therefore by a standard argument it is at most

$$\ll_{K,\varepsilon}\frac{(\mathcal{N}\gcd(\mathfrak{m},\mathfrak{c}))^{1/2}(\mathcal{N}\mathfrak{m})^{1/2+3\varepsilon}}{(\mathcal{N}(\mathfrak{c}\mathfrak{a}^{-1}))^{1+\varepsilon}}\,.$$

Altogether the off-diagonal term contributes

$$\ll_{K,\varepsilon} \Xi^{2+\varepsilon}\big(\mathcal{N}\gcd(\mathfrak{m},\mathfrak{c})\big)^{1/2}(\mathcal{N}\mathfrak{m})^{1/2+\varepsilon}. \tag{97}$$

□

3 Part II: Subconvexity and Shifted Convolution Sums

3.1 Heuristic explanation of the exponent. The Burgess exponent 3/8 for GL_2, or 3/16 for GL_1, seems to be a universal barrier, and there are several quite distinct methods that independently yield it (perhaps in a slightly weaker version coming from possible non-tempered representations). Therefore it might be instructive to sketch the subconvexity argument neglecting all the technical details to show where the exponents come from in our method. This is not intended to be a proof of any kind, but an experienced reader will have little difficulty in reconstructing a rigorous proof from the following remarks. For simplicity let us assume that $K=\mathbb{Q}$, and the conductor of π is 1, and let us also assume the Ramanujan–Petersson conjecture. Moreover we will not display epsilons.

We consider the amplified moment

$$\sum_{\omega\,(\mathrm{mod}\,q)}\Big|\sum_{\ell\sim L}\omega(\ell)\bar{\chi}(\ell)\Big|^2\big|L(1/2,\pi\otimes\omega)\big|^2.$$

On the one hand, this is

$$\gg L^2\big|L(1/2,\pi\otimes\chi)\big|^2, \tag{98}$$

on the other hand, this is

$$\ll q\sum_{\ell_1,\ell_2\sim L}\chi(\ell_1)\bar{\chi}(\ell_2)\sum_{\ell_1 m-\ell_2 n\equiv 0\,(\mathrm{mod}\,q)}\frac{\lambda_\pi(m)\bar{\lambda}_\pi(n)}{\sqrt{mn}}W\left(\frac{m}{q}\right)\bar{W}\left(\frac{n}{q}\right).$$

We single out the term $\ell_1 m-\ell_2 n=0$ which essentially implies $\ell_1=\ell_2$, $m=n$ and hence contributes

$$\ll qL. \tag{99}$$

We write the off-diagonal contribution of the inner sum as

$$\sum_{h\sim L}\sum_{\ell_1 m-\ell_2 n=qh}\frac{\lambda_\pi(m)\bar{\lambda}_\pi(n)}{\sqrt{mn}}W\left(\frac{m}{q}\right)\bar{W}\left(\frac{n}{q}\right). \tag{100}$$

By the surjectivity of the Kirillov map, we can find a vector $\phi\in V_\pi$ such that the inner sum is the horocycle integral

$$\int_0^1\underbrace{(R_{\ell_1}\phi R_{\ell_2}\bar{\phi})}_{=:\Phi}\left(\begin{pmatrix}(qL)^{-1} & x\\ 0 & 1\end{pmatrix}\right)e(-qhx)dx\,,$$

where R_ℓ is the shift operator (14). We decompose the form Φ (which is of level $\sim L^2$) spectrally (ignoring the continuous spectrum) as $\Phi=\sum_j\Phi_j$, so that we can recast (100) roughly as

$$\sum_{h\sim L}\sum_j\frac{\lambda_j(qh)}{\sqrt{qh}}W_{\Phi_j}\left(\frac{h}{L}\right) \tag{101}$$

with the notation (40)–(41). In particular, W_{Φ_j} is a multiple of the Whittaker function and therefore decays rapidly in the spectral parameter λ_j. By Plancherel and the fact that the Kirillov map is almost an isometry (see (42)), we have $\sum_j \|W_{\Phi_j}\|^2 \sim \|\Phi\| \ll 1$, since the operators R_ℓ are isometries. By Weyl's law, there are about L^2 eigenvalues in an interval of constant length, so the j-sum has effectively about L^2 terms, and hence each $W_{\Phi_j}(h/L) \sim W_{\Phi_j}(1)$ should be of size $1/L$. At this point we can already sum trivially to get an off-diagonal contribution of

$$q \underbrace{L^2}_{\text{amplifier}} \underbrace{L}_{h\text{-sum}} \underbrace{L^2}_{j\text{-sum}} \frac{1}{\sqrt{qL}} \frac{1}{L} = q^{1/2} L^{7/2}. \tag{102}$$

Combining (98), (99) and (102) gives $L(1/2, \pi \otimes \chi) \ll q^{2/5}$ upon choosing $L = q^{1/5}$.

However, we can do better by exploiting cancellation in the double sum over j and h. One way to see this is to recognize that the h-sum mimics the central value $L(1/2, \pi_j)$ (the length is L and the conductor is L^2), and on average over j we should be able to prove Lindelöf, that is, on average we should have $\sum_{h\sim L} \lambda_j(h) h^{-1/2} \sim 1$ rather than $L^{1/2}$. This can be made precise as follows: by Cauchy–Schwarz, (101) is bounded by

$$\frac{1}{\sqrt{q}L} \Big(\sum_{\lambda_j \sim 1} 1 \Big)^{1/2} \Bigg(\sum_{h_1, h_2 \sim L} \frac{1}{\sqrt{h_1 h_2}} \sum_{\lambda_j \sim 1} \lambda_j(h_1) \bar{\lambda}_j(h_2) \Bigg)^{1/2}. \tag{103}$$

The Kuznetsov formula translates the innermost sum into

$$L^2 \left(\delta_{h_1, h_2} + \sum_{L^2 \mid c} \frac{1}{c} S(h_1, h_2, c) f\left(\frac{h_1 h_2}{c^2} \right) \right),$$

where $S(h_1, h_2, c) \ll c^{1/2}$ and $f(x) \ll \min(1, x^{1/2})$, so that (103) is by Weil's bound and by trivial estimates $\ll L q^{-1/2}$, and hence the complete off-diagonal term is $\ll q^{1/2} L^3$. This yields $L(1/2, \pi \otimes \chi) \ll q^{3/8}$ upon choosing $L = q^{1/4}$.

3.2 Shifted convolution sums. In this section we will appeal to the spectral decomposition (8) for trivial ω. We will work with the subspace $L^2(\mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbb{A})/T\mathcal{K}(\mathfrak{c}), \mathrm{triv})$ which is also a component of the subspace FS according to (94). To simplify notation, we drop the subscripts ω in $\mathcal{C}_\omega(\mathfrak{c})$, $\mathcal{E}_\omega(\mathfrak{c})$ etc., and we use the abbreviations (cf. (22), (30), (67))

$$\int_{(\mathfrak{c})} f_\varpi \, d\varpi := \sum_{\substack{\pi \in \mathcal{C}(\mathfrak{c}) \\ \varepsilon_\pi = 1}} f_\pi + \int_{\substack{\varpi \in \mathcal{E}(\mathfrak{c}) \\ \varepsilon_\varpi = 1}} f_\varpi d\varpi$$

for any quantity f indexed by irreducible automorphic representations. The aim of this section is to prove the following central result.

Theorem 2. *Let π_1, π_2 be two irreducible cuspidal representations of $\mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbb{A})$ with the same unitary central character and signature character. Let $\ell_1, \ell_2 \in \mathfrak{o}$ be nonzero integers and write $\mathfrak{c} := \mathrm{lcm}(\ell_1 \mathfrak{c}_{\pi_1}, \ell_2 \mathfrak{c}_{\pi_2})$. Let $a, b, c \in \mathbb{N}_0$, and let $W_1, W_2 : K_\infty^\times \to \mathbb{C}$ be arbitrary functions such that $\|W_{1,2}\|_{A^\mu}$ given by (88) exist for $\mu := 2d(8 + a + b + 2c)$. Let $P \in \mathbb{C}[x_1, \dots, x_d]$ be a polynomial of degree at most a in each variable, and consider the differential operator*

$\mathcal{D} := P(y_1\partial_{y_1}, \dots, y_d\partial_{y_d})$. Then for any $\varpi \in \mathcal{C}(\mathfrak{c}) \cup \mathcal{E}(\mathfrak{c})$ with $\varepsilon_\varpi = 1$ and for any $\mathfrak{t} \mid \mathfrak{c}\mathfrak{c}_\varpi^{-1}$ there exists a function $W_{\varpi,\mathfrak{t}} : K_\infty^\times \to \mathbb{C}$ depending only on $\pi_{1,2}$, $W_{1,2}$, ϖ, $\mathfrak{t}$, K such that the following two properties hold.

- *For $Y \in (0,\infty)^d$, an ideal $\mathfrak{y} \subseteq \mathfrak{o}$ and a nonzero $q \in \mathfrak{y}$ there is a spectral decomposition*

$$\sum_{\substack{\ell_1 r_1 - \ell_2 r_2 = q \\ 0 \neq r_{1,2} \in \mathfrak{y}}} \frac{\lambda_{\pi_1}(r_1\mathfrak{y}^{-1})\bar{\lambda}_{\pi_2}(r_2\mathfrak{y}^{-1})}{\sqrt{\mathcal{N}(r_1 r_2 \mathfrak{y}^{-2})}} \cdot W_1\left(\frac{(\ell_1 r_1)^{\sigma_1}}{Y_1}, \dots, \frac{(\ell_1 r_1)^{\sigma_d}}{Y_d}\right) \bar{W}_2\left(\frac{(\ell_2 r_2)^{\sigma_1}}{Y_1}, \dots, \frac{(\ell_2 r_2)^{\sigma_d}}{Y_d}\right)$$
$$= \int_{(\mathfrak{c})} \sum_{\mathfrak{t} \mid \mathfrak{c}\mathfrak{c}_\varpi^{-1}} \frac{\lambda_\varpi^{(\mathfrak{t})}(q\mathfrak{y}^{-1})}{\sqrt{\mathcal{N}(q\mathfrak{y}^{-1})}} W_{\varpi,\mathfrak{t}}\left(\frac{q^{\sigma_1}}{Y_1}, \dots, \frac{q^{\sigma_d}}{Y_d}\right) d\varpi\,, \qquad (104)$$

  where $\lambda_\varpi^{(\mathfrak{t})}(\mathfrak{m})$ is given by (40) and (48).
- *For $y \in K_\infty^\times$, $0 < \varepsilon < 1/4$, and θ as in (11), there is a bound*

$$\int_{(\mathfrak{c})} \sum_{\mathfrak{t} \mid \mathfrak{c}\mathfrak{c}_\varpi^{-1}} (\mathcal{N}\tilde{\lambda}_\varpi)^{2c} \left|\mathcal{D}W_{\varpi,\mathfrak{t}}(y)\right|^2 d\varpi \ll \left|\mathcal{N}(\ell_1\ell_2)\right|^\varepsilon \|W_1\|_{A^\mu}^2 \|W_2\|_{A^\mu}^2 \prod_{j=1}^d |y_j|^{1-2\theta-\varepsilon} \min\left(1, |y_j|^{-2b}\right)$$

  with an implied constant depending only on $\pi_{1,2}$, a, b, c, P, K, ε.

Remark 11. For $q \notin \mathfrak{y}$ the left-hand side of (104) vanishes trivially. The assumptions on $\pi_{1,2}$ only serve notational convenience, and with a little more work one can show that the implied constant depends polynomially on $C(\pi_1)C(\pi_2)$.

Remark 12. One can combine the L^2-bound in Theorem 2 for $c+1$ in place of c with Cauchy–Schwarz and Lemma 6 (resp. Lemma 2) to deduce an L^1-bound for the cuspidal (resp. continuous) spectrum. For $\kappa := 2d(10+a+b+2c)$ one obtains

$$\int_{\substack{\varpi \in \mathcal{C}(\mathfrak{c}) \\ \varepsilon_\varpi = 1}} \sum_{\mathfrak{t} \mid \mathfrak{c}\mathfrak{c}_\varpi^{-1}} (\mathcal{N}\tilde{\lambda}_\varpi)^{c} \left|\mathcal{D}W_{\varpi,\mathfrak{t}}(y)\right| d\varpi \ll \left|\mathcal{N}(\ell_1\ell_2)\right|^{\frac{1}{2}+\varepsilon} \|W_1\|_{A^\kappa} \|W_2\|_{A^\kappa} \prod_{j=1}^d |y_j|^{\frac{1}{2}-\theta-\varepsilon} \min\left(1, |y_j|^{-b}\right)$$

and, denoting by $\mathfrak{l} \subseteq \mathfrak{o}$ the largest *square* divisor of $\operatorname{lcm}((\ell_1),(\ell_2))$,

$$\int_{\substack{\varpi \in \mathcal{E}(\mathfrak{c}) \\ \varepsilon_\varpi = 1}} \sum_{\mathfrak{t} \mid \mathfrak{c}\mathfrak{c}_\varpi^{-1}} (\mathcal{N}\tilde{\lambda}_\varpi)^{c} \left|\mathcal{D}W_{\varpi,\mathfrak{t}}(y)\right| d\varpi \ll (\mathcal{N}\mathfrak{l})^{\frac{1}{4}} \left|\mathcal{N}(\ell_1\ell_2)\right|^{\varepsilon} \|W_1\|_{A^\kappa} \|W_2\|_{A^\kappa} \prod_{j=1}^d |y_j|^{\frac{1}{2}-\theta-\varepsilon} \min\left(1, |y_j|^{-b}\right),$$

with implied constants depending only on $\pi_{1,2}$, a, b, c, P, K, ε.

Proof. By the surjectivity of the Kirillov map (see the remark after (37)) we can choose $\phi_i \in V_{\pi_i}(\mathfrak{c}_{\pi_i})$ for $i = 1, 2$, such that $W_{\phi_i} = W_i$. Let

$$\Phi := \left(R_{(\ell_1)}\phi_1\right)\left(R_{(\ell_2)}\bar{\phi}_2\right)$$

with the notation as in (14). Then $\Phi \in L^2(\mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbb{A})/T\mathcal{K}(\mathfrak{c}), \mathrm{triv})$ with $\mathfrak{c}$ as in the theorem. Let $y \in \mathbb{A}^\times$ be such that $y_\infty = (Y_1, \dots, Y_d)$ and $(y_{\mathrm{fin}}) = \mathfrak{y}$. By (14) and (34) we have

$$\sum_{\substack{\ell_1 r_1 - \ell_2 r_2 = q \\ 0 \neq r_{1,2} \in \mathfrak{y}}} \frac{\lambda_{\pi_1}(r_1\mathfrak{y}^{-1})\bar{\lambda}_{\pi_2}(r_2\mathfrak{y}^{-1})}{\sqrt{\mathcal{N}(r_1 r_2 \mathfrak{y}^{-2})}}$$

$$\cdot W_1\left(\frac{(\ell_1 r_1)^{\sigma_1}}{Y_1}, \dots, \frac{(\ell_1 r_1)^{\sigma_d}}{Y_d}\right) \bar{W}_2\left(\frac{(\ell_2 r_2)^{\sigma_1}}{Y_1}, \dots, \frac{(\ell_2 r_2)^{\sigma_d}}{Y_d}\right)$$

$$= \int_{K\backslash\mathbb{A}} \Phi\left(\begin{pmatrix} y^{-1} & x \\ 0 & 1 \end{pmatrix}\right) \psi(-qx)dx \,.$$

By (8), (39), (47), we have an orthogonal decomposition

$$\Phi = \Phi_{\mathrm{sp}} + \int_{(\mathfrak{c})} \sum_{\mathfrak{t} | \mathfrak{c}\mathfrak{c}_\varpi^{-1}} \Phi_{\varpi,\mathfrak{t}} d\varpi \,, \quad \Phi_{\mathrm{sp}} \in L_{\mathrm{sp}} \,, \ \Phi_{\varpi,\mathfrak{t}} \in R^{(\mathfrak{t})} V_\varpi(\mathfrak{c}_\varpi) \,,$$

where Φ_{sp} is the projection of Φ on the subspace generated by the functions $g \mapsto \chi(\det g)$ with any quadratic Hecke character χ as discussed in section 2.2. As q is nonzero, (104) is immediate from (40) and (48) upon defining $W_{\varpi,\mathfrak{t}} := W_{\Phi_{\varpi,\mathfrak{t}}}$.

We proceed to establish the upper bound stated in the theorem. By Lemma 4 and (85) we have

$$\int_{(\mathfrak{c})} \sum_{\mathfrak{t} | \mathfrak{c}\mathfrak{c}_\varpi^{-1}} (\mathcal{N}\tilde{\lambda}_\varpi)^{2c} \left|\mathcal{D}W_{\varpi,\mathfrak{t}}(y)\right|^2 d\varpi \ll (\mathcal{N}\mathfrak{c})^\varepsilon \|\Phi\|_{S^\alpha}^2 \prod_{j=1}^d |y_j|^{1-2\theta-\varepsilon} \min\left(1, |y_j|^{-b}\right),$$

where $\alpha := d(5 + a + b + 2c)$. It remains to show that

$$\|\Phi\|_{S^\alpha} \ll \|W_1\|_{A^\mu} \|W_2\|_{A^\mu} \tag{105}$$

for μ as in the theorem. By Lemma 5 any $\mathcal{D} \in U(\mathfrak{g})$ of order at most α satisfies

$$\left\|\mathcal{D}R_{(\ell_i)}\phi_i\right\|_\infty = \left\|R_{(\ell_i)}\mathcal{D}\phi_i\right\|_\infty = \|\mathcal{D}\phi_i\|_\infty \ll_{\pi_i,K} \|\phi_i\|_{S^{\alpha+3d}} \,,$$

therefore the Leibniz rule for derivations immediately shows

$$\|\Phi\|_{S^\alpha} \ll_{\pi_1,\pi_2,K} \|\phi_1\|_{S^{\alpha+3d}} \|\phi_2\|_{S^{\alpha+3d}} \,.$$

An application of Lemma 3 and (87) now yields (105) and completes the proof of Theorem 2. □

3.3 A Burgess-like subconvex bound for twisted L-functions. In this section we prove Theorem 1, borrowing several important ideas from [CoPS], [Co]. For simplicity, we shall in general not indicate the dependence of implied constants on π, χ_∞, K. We regard χ as a Grössencharacter, i.e. a certain character of the group of fractional ideals coprime to $\mathfrak{q}$. We extend χ to the group of all fractional ideals by defining it to be zero for fractional ideals not coprime to $\mathfrak{q}$. There exists a pair of characters $\chi_{\mathrm{fin}} : (\mathfrak{o}/\mathfrak{q})^\times \to S^1$ and $\chi_\infty : K_\infty^\times \to S^1$ such that $\chi((r)) = \chi_{\mathrm{fin}}(r)\chi_\infty(r)$ for $r \in \mathfrak{o}$ coprime to $\mathfrak{q}$. We lift any character ξ of $(\mathfrak{o}/\mathfrak{q})^\times$ to a function $\xi : \mathfrak{o} \to \mathbb{C}$ by defining $\xi(r) = \xi(r \bmod \mathfrak{q})$ for $r \in \mathfrak{o}$ coprime to $\mathfrak{q}$ and $\xi(r) = 0$ elsewhere.

Our starting point is the approximate functional equation in the user-friendly version (75). We cut the sum into (finitely many) pieces according to the narrow ideal class of the ideal $\mathfrak{m}$. We fix a narrow ideal class and a representative $\mathfrak{y}$ coprime to $\mathfrak{q}$; we can assume $\mathcal{N}\mathfrak{y} \ll_\varepsilon (\mathcal{N}\mathfrak{q})^\varepsilon$. Then it is enough to bound

$$\sum_{\substack{0<<r\in\mathfrak{y} \\ r \bmod U^+}} \frac{\lambda_\pi(r\mathfrak{y}^{-1})\chi(r\mathfrak{y}^{-1})}{\sqrt{\mathcal{N}(r\mathfrak{y}^{-1})}} V\left(\frac{\mathcal{N}r}{Y}\right) \tag{106}$$

for $Y \ll_\varepsilon (\mathcal{N}\mathfrak{q})^{1+\varepsilon}$ and a smooth function $V:(0,\infty)\to\mathbb{C}$ supported on $[1/2,2]$ such that $V^{(j)}(y) \ll_j 1$ for all $j\in\mathbb{N}_0$. Let us fix (once and for all) a fundamental domain $\mathcal{F}_0$ for the action of U^+ on the hyperboloid $\{y\in K_{\infty,+}^\times \mid \mathcal{N}y=1\}$ such that its image under the map $K_{\infty,+}^\times\to\mathbb{R}^d$, $y\mapsto(\log y^{\sigma_1},\dots,\log y^{\sigma_d})$, is a fundamental parallelotope of the image of U^+ under the same map. (The image of U^+ is a lattice in the hyperplane of K_∞ orthogonal to $(1,\dots,1)$.) The cone $\mathcal{F}:=K_{\infty,+}^{\mathrm{diag}}\mathcal{F}_0$ is a fundamental domain for the action of U^+ on $K_{\infty,+}^\times$. We introduce the following smooth variants of $\mathcal{F}_0$ and $\mathcal{F}$: we fix a smooth and compactly supported function $F_0:\{y\in K_{\infty,+}^\times \mid \mathcal{N}y=1\}\to\mathbb{C}$ such that $\sum_{u\in U^+}F_0(uy)=1$ for any $y\in K_{\infty,+}^\times$ of norm 1, and we extend this to all of $K_{\infty,+}^\times$ by $F(y):=F_0(y/(\mathcal{N}y)^{1/d})$. Note that the support of F_0 is contained in some box $[c_1,c_2]^d\subseteq K_{\infty,+}^\times$, and the support of F is contained in the cone $K_{\infty,+}^{\mathrm{diag}}[c_1,c_2]^d$ of this box. We can rewrite (106) as

$$\sum_{0<<r\in\mathfrak{y}} \frac{\lambda_\pi(r\mathfrak{y}^{-1})\chi(r\mathfrak{y}^{-1})}{\sqrt{\mathcal{N}(r\mathfrak{y}^{-1})}} F(r) V\left(\frac{\mathcal{N}r}{Y}\right)$$

which is really a finite sum, because $\mathfrak{y}$ is a lattice in K_∞ and the terms vanish outside the box $\left[\frac{1}{2}c_1Y^{1/d}, 2c_2Y^{1/d}\right]^d$. Let us fix a smooth function $W:K_{\infty,+}^\times\to\mathbb{C}$ supported on $\left[\frac{1}{3}c_1,3c_2\right]^d$ such that $W(y)=1$ on $\left[\frac{1}{2}c_1,2c_2\right]^d$, then we can recast (106) as

$$\begin{aligned}
&\bar\chi(\mathfrak{y})\sum_{0<<r\in\mathfrak{y}} \frac{\lambda_\pi(r\mathfrak{y}^{-1})\chi((r))}{\sqrt{\mathcal{N}(r\mathfrak{y}^{-1})}} F(r) V\left(\frac{\mathcal{N}r}{Y}\right) W\left(\frac{r}{Y^{1/d}}\right) \\
&\qquad = \frac{\bar\chi(\mathfrak{y})\chi_\infty(Y^{1/d})}{(2\pi i)^d}\int_{(i\mathbb{R})^d} \check V(v) \sum_{0<<r\in\mathfrak{y}} \frac{\lambda_\pi(r\mathfrak{y}^{-1})\chi_{\mathrm{fin}}(r)}{\sqrt{\mathcal{N}(r\mathfrak{y}^{-1})}} W_v\left(\frac{r}{Y^{1/d}}\right) dv\,,
\end{aligned} \tag{107}$$

where $v:=(v_1,\dots,v_d)\in(i\mathbb{R})^d$ and

$$\check V(v):=\int_{K_{\infty,+}^\times} F(y)V(\mathcal{N}y)\chi_\infty(y)\prod_{j=1}^d y_j^{v_j}\,d^\times y\,,\quad W_v(y):=W(y)\prod_{j=1}^d y_j^{-v_j}.$$

At this point it is worthwhile to extend the notational convention in (12) to all complex vectors $z\in\mathbb{C}^d$ as follows:

$$\tilde z := \left(1+|z_j|\right)_{j=1}^d \in \mathbb{R}_{>0}^d\,. \tag{108}$$

The functions $F(y)V(\mathcal{N}y)$ and $W(y)$ are smooth of compact support and $\chi_\infty(y)=\prod_{j=1}^d y_j^{s_j}$ for some fixed $s\in(i\mathbb{R})^d$, therefore we have the bounds

$$\check V(v) \ll_{A,\chi_\infty} (\mathcal{N}\tilde v)^{-A}\,,\quad A>0\,, \tag{109}$$

$$\partial_{y_1}^{\mu_1}\cdots\partial_{y_d}^{\mu_d} W_v(y) \ll_\mu \prod_{j=1}^d \left(1+|v_j|\right)^{\mu_j},\quad \mu\in\mathbb{N}_0^d\,. \tag{110}$$

We fix $v \in (i\mathbb{R})^d$ and postpone the integration over v to the very end of the argument. For a character ξ of $(\mathfrak{o}/\mathfrak{q})^\times$ we define

$$\mathcal{L}_\xi(v) := \sum_{0<<r\in\mathfrak{y}} \frac{\lambda_\pi(r\mathfrak{y}^{-1})\xi(r)}{\sqrt{\mathcal{N}(r\mathfrak{y}^{-1})}} W_v\left(\frac{r}{Y^{1/d}}\right),$$

so that $\mathcal{L}_{\chi_{\rm fin}}(v)$ is the sum on the right-hand side of (107). Observe that the sum is supported in the box $\left[\frac{1}{3}c_1Y^{1/d}, 3c_2Y^{1/d}\right]^d$ whose cone $\mathcal{C} \subseteq K_{\infty,+}^\times$ is independent of Y and can be covered by finitely many U^+-translates of $\mathcal{F}$. We consider an amplified second moment and choose a parameter L satisfying $\log L \asymp \log(\mathcal{N}\mathfrak{q})$. It is not hard to see that

$$\#\left\{\mathfrak{l} \subseteq \mathfrak{o} \text{ is a totally positive principal prime ideal} \mid N\mathfrak{l} \in [L, 2L],\ \mathfrak{l} \nmid \mathfrak{q}\right\} \gg_\varepsilon L(N\mathfrak{q})^{-\varepsilon},$$

hence by positivity

$$\left|\mathcal{L}_{\chi_{\rm fin}}(v)\right|^2 \ll_\varepsilon \frac{(N\mathfrak{q})^\varepsilon}{L^2} \sum_{\xi\in\widehat{(\mathfrak{o}/\mathfrak{q})^\times}} \left|\mathcal{L}_\xi(v) \sum_{\substack{\ell\in\mathfrak{o}\cap\mathcal{F}\\ \mathcal{N}\ell\in[L,2L]\\ (\ell)\text{ prime}, (\ell)\nmid\mathfrak{q}}} \xi(\ell)\bar{\chi}_{\rm fin}(\ell)\right|^2.$$

By Plancherel's formula for $(\mathfrak{o}/\mathfrak{q})^\times$ this is the same as

$$\left|\mathcal{L}_{\chi_{\rm fin}}(v)\right|^2 \ll_\varepsilon \frac{\varphi(\mathfrak{q})(\mathcal{N}\mathfrak{q})^\varepsilon}{L^2} \sum_{x\in(\mathfrak{o}/\mathfrak{q})^\times} \left| \sum_{\substack{\ell\in\mathfrak{o}\cap\mathcal{F}\\ \mathcal{N}\ell\in[L,2L]\\ (\ell)\text{ prime}, (\ell)\nmid\mathfrak{q}}} \bar{\chi}_{\rm fin}(\ell) \sum_{\substack{r\in\mathfrak{y}\cap\mathcal{C}\\ \ell r\equiv x \,(\mathrm{mod}\ \mathfrak{q})}} \frac{\lambda_\pi(r\mathfrak{y}^{-1})}{\sqrt{\mathcal{N}(r\mathfrak{y}^{-1})}} W_v\left(\frac{r}{Y^{1/d}}\right)\right|^2.$$

We can extend the summation over all $x \in \mathfrak{o}/\mathfrak{q}$ by positivity, then after opening the square we get

$$\begin{aligned}\left|\mathcal{L}_{\chi_{\rm fin}}(v)\right|^2 \ll_\varepsilon\ & \frac{(\mathcal{N}\mathfrak{q})^{1+\varepsilon}}{L^2} \sum_{\substack{\ell_1,\ell_2\in\mathfrak{o}\cap\mathcal{F}\\ \mathcal{N}\ell_1,\mathcal{N}\ell_2\in[L,2L]\\ (\ell_1),(\ell_2)\text{ primes}\\ (\ell_1),(\ell_2)\nmid\mathfrak{q}}} \bar{\chi}(\ell_1)\chi(\ell_2)\\ &\times \sum_{\substack{\ell_1r_1-\ell_2r_2\in\mathfrak{q}\\ r_1,r_2\in\mathfrak{y}\cap\mathcal{C}}} \frac{\lambda_\pi(r_1\mathfrak{y}^{-1})\bar{\lambda}_\pi(r_2\mathfrak{y}^{-1})}{\sqrt{\mathcal{N}(r_1r_2\mathfrak{y}^{-2})}} W_v\left(\frac{r_1}{Y^{1/d}}\right)\bar{W}_v\left(\frac{r_2}{Y^{1/d}}\right). \end{aligned}\tag{111}$$

We single out the diagonal term $\ell_1r_1 - \ell_2r_2 = 0$ which contributes at most

$$\ll_\varepsilon \frac{(\mathcal{N}\mathfrak{q})^{1+\varepsilon}}{L^2} \sum_{\substack{\ell\in\mathfrak{o}\cap\mathcal{F}\\ \mathcal{N}\ell\asymp L}} \sum_{\substack{r\in\mathfrak{y}\cap\mathcal{C}\\ \mathcal{N}r\asymp Y}} \frac{|\lambda_\pi(r\mathfrak{y}^{-1})|^2}{\mathcal{N}(r\mathfrak{y}^{-1})} \#\left\{(\ell', r') \in (\mathfrak{o}\cap\mathcal{F}) \times (\mathfrak{y}\cap\mathcal{C}) \mid \ell'r' = \ell r\right\},$$

uniformly in $v \in (i\mathbb{R})^d$. The last factor is bounded by $\ll_\varepsilon (LY)^\varepsilon$, and so by (77) the preceding display is at most

$$\ll_\varepsilon \frac{(\mathcal{N}\mathfrak{q})^{1+\varepsilon}}{L^2} \#\{\mathfrak{l} \subseteq \mathfrak{o} \mid \mathcal{N}\mathfrak{l} \asymp L\} \sum_{\mathcal{N}\mathfrak{m}\ll Y} \frac{|\lambda_\pi(\mathfrak{m})|^2}{\mathcal{N}\mathfrak{m}} \ll_\varepsilon (\mathcal{N}\mathfrak{q})^\varepsilon \frac{\mathcal{N}\mathfrak{q}}{L}. \tag{112}$$

Let us now consider the off-diagonal contribution in (111). If $[c_3, c_4]^d \subseteq K_{\infty,+}^\times$ is a box containing $\mathcal{F}_0$, then in (111) the variables satisfy $\ell_{1,2} \in [c_3L^{1/d}, 2c_4L^{1/d}]^d$ and

$r_{1,2} \in \left[\frac{1}{3}c_1 Y^{1/d}, 3c_2 Y^{1/d}\right]^d$, so that

$$\ell_1 r_1 - \ell_2 r_2 \in \mathcal{B} := \left[-6c_2c_4(LY)^{1/d}, 6c_2c_4(LY)^{1/d}\right]^d.$$

We fix $\ell_{1,2}$ for the moment and we rewrite the off-diagonal part of the inner sum in (111) as

$$\sum_{0\neq q\in\mathfrak{q}\mathfrak{y}\cap\mathcal{B}} \sum_{\substack{\ell_1r_1-\ell_2r_2=q\\ 0\neq r_1,r_2\in\mathfrak{y}}} \frac{\lambda_\pi(r_1\mathfrak{y}^{-1})\bar{\lambda}_\pi(r_2\mathfrak{y}^{-1})}{\sqrt{\mathcal{N}(r_1r_2\mathfrak{y}^{-2})}} W_1\left(\frac{\ell_1r_1}{(LY)^{1/d}};v\right)\bar{W}_2\left(\frac{\ell_2r_2}{(LY)^{1/d}};v\right), \tag{113}$$

where $W_i(\cdot;v) : K_\infty^\times \to \mathbb{C}$ for $i = 1, 2$ are smooth functions defined by

$$W_i(y;v) := \begin{cases} W_v(\ell_i^{-1}L^{1/d}y), & y \in K_{\infty,+}^\times, \\ 0, & \text{otherwise}. \end{cases}$$

Note that $W_i(\cdot;v)$ is supported on $\left[\frac{1}{3}c_1c_3, 6c_2c_4\right]^d$ and by (110) it satisfies

$$\partial_{y_1}^{\mu_1}\cdots\partial_{y_d}^{\mu_d} W_i(y;v) \ll_\mu \prod_{j=1}^d \left(1+|v_j|\right)^{\mu_j}, \quad \mu\in\mathbb{N}_0^d. \tag{114}$$

Using Theorem 2, we rewrite (113) as

$$\sum_{0\neq q\in\mathfrak{q}\mathfrak{y}\cap\mathcal{B}} \int_{(\mathfrak{c})} \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}} \frac{\lambda_\varpi^{(\mathfrak{t})}(q\mathfrak{y}^{-1})}{\sqrt{\mathcal{N}(q\mathfrak{y}^{-1})}} W_{\varpi,\mathfrak{t}}\left(\frac{q}{(LY)^{1/d}};v\right) d\varpi, \tag{115}$$

where $\mathfrak{c} := \mathfrak{c}_\pi \operatorname{lcm}((\ell_1),(\ell_2))$. At this point we can already estimate the Eisenstein contribution trivially. On the one hand, we can combine the second bound in Remark 12 with (88) and (114) to see that (cf. (108))

$$\int_{\substack{\varpi\in\mathcal{E}(\mathfrak{c})\\ \varepsilon_\varpi=1}} \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}} \left|W_{\varpi,\mathfrak{t}}(y;v)\right| d\varpi \ll_\varepsilon \left(\mathcal{N}(\ell_1\ell_2)\right)^\varepsilon (\mathcal{N}\tilde{v})^{44d},$$

uniformly in $y\in K_\infty^\times$, $v\in(i\mathbb{R})^d$, $\ell_{1,2}$ and $\mathfrak{y}$. On the other hand, by (49) we have the uniform bound

$$\lambda_\varpi^{(\mathfrak{t})}(q\mathfrak{y}^{-1}) \ll_\varepsilon \left(\mathcal{N}\gcd(\mathfrak{c},(q))\right)(\mathcal{N}(q))^\varepsilon,$$

hence the Eisenstein contribution in (115) is at most

$$\ll_\varepsilon (\mathcal{N}\tilde{v})^{44d}(\mathcal{N}\mathfrak{q})^\varepsilon \sum_{0\neq q\in\mathfrak{q}\cap\mathcal{B}} \frac{\mathcal{N}\gcd(\mathfrak{c},(q))}{\sqrt{\mathcal{N}(q)}}.$$

In the last sum each principal ideal (q) has multiplicity $\ll (\log(\mathcal{N}\mathfrak{q}))^{d-1}$. Indeed, any nonzero principal ideal in $\mathfrak{o}$ has a generator q such that $|q^{\sigma_j}| \geqslant c_3$ for $1\leqslant j\leqslant d$, so the multiplicity in question is at most the number of units $u\in U$ in the cube $[-c_5(LY)^{1/d}, c_5(LY)^{1/d}]^d$ for $c_5 := 6c_2c_4/c_3$ which is $\ll (\log(\mathcal{N}\mathfrak{q}))^{d-1}$ by Dirichlet's unit theorem (or its proof). At any rate, the last sum is

$$\ll_\varepsilon (\mathcal{N}\mathfrak{q})^\varepsilon \sum_{\substack{(q)\subseteq\mathfrak{q}\\ \mathcal{N}(q)\ll LY}} \frac{\mathcal{N}\gcd(\mathfrak{c},(q))}{\sqrt{\mathcal{N}(q)}} \ll_\varepsilon (\mathcal{N}\mathfrak{q})^{-1+2\varepsilon}(LY)^{1/2} \ll_\varepsilon (\mathcal{N}\mathfrak{q})^{-1/2+3\varepsilon}L^{1/2},$$

hence the Eisenstein contribution in (115) is at most

$$\ll_\varepsilon (\mathcal{N}\tilde{v})^{44d}(\mathcal{N}\mathfrak{q})^{-1/2+\varepsilon}L^{1/2}. \tag{116}$$

Let us now turn to the cuspidal contribution in (115). Choosing $a=0$, $b=1$, c very large in Theorem 2 and combining the inequality there with (88) and (114), Cauchy–Schwarz and Lemma 6, we see for any $\varepsilon>0$ that the contribution of all $\varpi\in\mathcal{C}(\mathfrak{c})$ with $\tilde{\lambda}_{\varpi,j}\geqslant(\mathcal{N}\mathfrak{q})^\varepsilon$ for some j is negligible. Let us introduce the notation

$$\mathcal{C}(\mathfrak{c},\varepsilon):=\left\{\varpi\in\mathcal{C}(\mathfrak{c})\mid\tilde{\lambda}_{\varpi,j}\leqslant(\mathcal{N}\mathfrak{q})^\varepsilon\text{ for }1\leqslant j\leqslant d\right\},\quad \varepsilon>0\,,$$

and

$$\mathcal{B}(\xi):=\left\{y\in\mathcal{B}\mid\operatorname{sgn}(y)=\xi\right\},\quad \xi\in\{\pm1\}^d.$$

Then it suffices to bound, for fixed primes $\ell_1,\ell_2>>0$ and $\varepsilon>0$, $\xi\in\{\pm1\}^d$ the quantity

$$\sum_{q\in\mathfrak{q}\mathfrak{y}\cap\mathcal{B}(\xi)}\sum_{\substack{\varpi\in\mathcal{C}(\mathfrak{c},\varepsilon)\\ \mathfrak{t}\mid\mathfrak{c}\mathfrak{c}_\varpi^{-1}}}\frac{\lambda_\varpi^{(\mathfrak{t})}(q\mathfrak{y}^{-1})}{\sqrt{\mathcal{N}(q\mathfrak{y}^{-1})}}W_{\varpi,\mathfrak{t}}\left(\frac{q}{(LY)^{1/d}};v\right). \tag{117}$$

We separate the variables ϖ and q by Mellin inversion. For $s\in\mathbb{C}^d$ with $\Re s_j\geqslant-1/4$ we write

$$\widehat{W}_{\varpi,\mathfrak{t}}^{(\xi)}(s;v):=\int_{K_{\infty,+}^\times}W_{\varpi,\mathfrak{t}}(\xi y;v)\prod_{j=1}^d y_j^{s_j}\,d^\times y$$

and recast (117) as

$$\left(\frac{1}{2\pi i}\right)^d\int_{(i\mathbb{R})^d}(LY)^{(s_1+\cdots+s_d)/d}\sum_{\substack{\varpi\in\mathcal{C}(\mathfrak{c},\varepsilon)\\ \mathfrak{t}\mid\mathfrak{c}\mathfrak{c}_\varpi^{-1}}}\widehat{W}_{\varpi,\mathfrak{t}}^{(\xi)}(s;v)\sum_{q\in\mathfrak{q}\mathfrak{y}\cap\mathcal{B}(\xi)}\frac{\lambda_\varpi^{(\mathfrak{t})}(q\mathfrak{y}^{-1})}{\sqrt{\mathcal{N}(q\mathfrak{y}^{-1})}}\prod_{j=1}^d|q^{\sigma_j}|^{-s_j}ds$$

$$\ll\int_{(i\mathbb{R})^d}\Bigg(\sum_{\substack{\varpi\in\mathcal{C}(\mathfrak{c},\varepsilon)\\ \mathfrak{t}\mid\mathfrak{c}\mathfrak{c}_\varpi^{-1}}}\left|\widehat{W}_{\varpi,\mathfrak{t}}^{(\xi)}(s;v)\right|^2\Bigg)^{\frac12}\Bigg(\sum_{\substack{\varpi\in\mathcal{C}(\mathfrak{c},\varepsilon)\\ \mathfrak{t}\mid\mathfrak{c}\mathfrak{c}_\varpi^{-1}}}\Bigg|\sum_{q\in\mathfrak{q}\mathfrak{y}\cap\mathcal{B}(\xi)}\frac{\lambda_\varpi^{(\mathfrak{t})}(q\mathfrak{y}^{-1})}{\sqrt{\mathcal{N}(q\mathfrak{y}^{-1})}}\prod_{j=1}^d|q^{\sigma_j}|^{-s_j}\Bigg|^2\Bigg)^{\frac12}|ds|\,.$$

Using the differential operator $\mathcal{D}:=\prod_{j=1}^d(1+y_j\partial_{y_j})^3$ the first sum is, for any $s\in(i\mathbb{R})^d$,

$$\ll(\mathcal{N}\tilde{s})^{-3}\int_{K_\infty^\times}\int_{K_\infty^\times}\sum_{\substack{\varpi\in\mathcal{C}(\mathfrak{c},\varepsilon)\\ \mathfrak{t}\mid\mathfrak{c}\mathfrak{c}_\varpi^{-1}}}\left|\mathcal{D}W_{\varpi,\mathfrak{t}}(y;v)\right|\left|W_{\varpi,\mathfrak{t}}(z;v)\right|d^\times y\,d^\times z\,,$$

$$\ll(\mathcal{N}\tilde{s})^{-3}\int_{K_\infty^\times}\int_{K_\infty^\times}\Bigg(\sum_{\substack{\varpi\in\mathcal{C}(\mathfrak{c},\varepsilon)\\ \mathfrak{t}\mid\mathfrak{c}\mathfrak{c}_\varpi^{-1}}}\left|\mathcal{D}W_{\varpi,\mathfrak{t}}(y;v)\right|^2\Bigg)^{1/2}\Bigg(\sum_{\substack{\varpi\in\mathcal{C}(\mathfrak{c},\varepsilon)\\ \mathfrak{t}\mid\mathfrak{c}\mathfrak{c}_\varpi^{-1}}}\left|W_{\varpi,\mathfrak{t}}(z;v)\right|^2\Bigg)^{1/2}d^\times y\,d^\times z\,.$$

We apply Theorem 2 with $(a,b,c)=(3,1,0)$ and $(a,b,c)=(0,1,0)$, then by (88) and (114) the integrand is

$$\ll_\varepsilon(\mathcal{N}q)^\varepsilon(\mathcal{N}\tilde{v})^{84d}\prod_{j=1}^d\min\left(|y_j|^{1/4},|y_j|^{-1/2}\right)\min\left(|z_j|^{1/4},|z_j|^{-1/2}\right),$$

so that the previous display is

$$\ll_\varepsilon (\mathcal{N}q)^\varepsilon (\mathcal{N}\tilde{v})^{84d}(\mathcal{N}\tilde{s})^{-3}.$$

We infer that (117) is bounded by

$$\ll_\varepsilon (\mathcal{N}q)^\varepsilon (\mathcal{N}\tilde{v})^{42d} \sup_{s\in(i\mathbb{R})^d} \Bigg(\sum_{\substack{\varpi\in\mathcal{C}(\mathfrak{c},\varepsilon)\\ \mathfrak{t}\mid\mathfrak{c}\mathfrak{c}_\varpi^{-1}}} \Bigg| \sum_{q\in\mathfrak{q}\mathfrak{y}\cap\mathcal{B}(\xi)} \frac{\lambda_\varpi^{(\mathfrak{t})}(q\mathfrak{y}^{-1})}{\sqrt{\mathcal{N}(q\mathfrak{y}^{-1})}} \prod_{j=1}^d |q^{\sigma_j}|^{-s_j} \Bigg|^2 \Bigg)^{1/2}. \tag{118}$$

Let us write for any nonzero ideal $\mathfrak{a}\subseteq\mathfrak{o}$ and any $s\in(i\mathbb{R})^d$

$$f(\mathfrak{a};s) := \sum_{\substack{q\in\mathcal{B}(\xi)\\ (q)=\mathfrak{a}\mathfrak{y}}} \prod_{j=1}^d |q^{\sigma_j}|^{-s_j},$$

then similarly as in the proof of (116) we have

$$\big|f(\mathfrak{a};s)\big| \leqslant \#\big\{q\in\mathcal{B} \mid (q)=\mathfrak{a}\mathfrak{y}\big\} \ll \big(\log(\mathcal{N}\mathfrak{q})\big)^{d-1} \ll_\varepsilon (\mathcal{N}\mathfrak{q})^\varepsilon, \tag{119}$$

while the q-sum in (118) equals the following sum over integral ideals $\mathfrak{m}$:

$$\sum_{\mathcal{N}\mathfrak{m}\ll LY/\mathcal{N}(\mathfrak{q}\mathfrak{y})} \frac{\lambda_\varpi^{(\mathfrak{t})}(\mathfrak{m}\mathfrak{q})}{\sqrt{\mathcal{N}(\mathfrak{m}\mathfrak{q})}} f(\mathfrak{m}\mathfrak{q};s)\,.$$

We now need to "factor out" $\lambda_\varpi^{(\mathfrak{t})}(\mathfrak{q})$. This is completely elementary, but a little tricky. First we rewrite the previous expression as

$$\sum_{\mathfrak{q}\mid\mathfrak{q}'\mid\mathfrak{q}^\infty} \sum_{\substack{\mathcal{N}\mathfrak{m}\ll LY/\mathcal{N}(\mathfrak{q}'\mathfrak{y})\\ \gcd(\mathfrak{m},\mathfrak{q})=\mathfrak{o}}} \frac{\lambda_\varpi^{(\mathfrak{t})}(\mathfrak{m}\mathfrak{q}')}{\sqrt{\mathcal{N}(\mathfrak{m}\mathfrak{q}')}} f(\mathfrak{m}\mathfrak{q}';s)\,.$$

Using the construction of $\lambda_\varpi^{(\mathfrak{t})}(\mathfrak{m})$ as given in (41) and the preceding remarks, together with the Hecke relation (31), we proceed as in [BlHM, p. 73–74] to show that

$$\lambda_\varpi^{(\mathfrak{t})}(\mathfrak{m}\mathfrak{q}') = \sum_{\mathfrak{b}\mid\gcd(\mathfrak{c},\mathfrak{q}',\frac{\mathfrak{q}'}{\gcd(\mathfrak{c},\mathfrak{q}')})} \mu(\mathfrak{b})\lambda_\varpi\left(\frac{\mathfrak{q}'}{\mathfrak{b}\gcd(\mathfrak{c},\mathfrak{q}')}\right)\lambda_\varpi^{(\mathfrak{t})}\left(\frac{\mathfrak{m}\gcd(\mathfrak{c},\mathfrak{q}')}{\mathfrak{b}}\right).$$

Since $\gcd(\mathfrak{c},\mathfrak{q}')$ divides $\mathfrak{c}_\pi$ (where π is the representation whose L-function we want to estimate), we can bound the q-sum in (118) by

$$\ll_\varepsilon \sum_{\mathfrak{q}\mid\mathfrak{q}'\mid\mathfrak{q}^\infty} (\mathcal{N}\mathfrak{q}')^{-1/2+\theta+\varepsilon} \sum_{\mathfrak{b}\mid\mathfrak{c}_\pi} \Bigg| \sum_{\substack{\mathcal{N}\mathfrak{m}\ll LY/\mathcal{N}(\mathfrak{q}'\mathfrak{y})\\ \gcd(\mathfrak{m},\mathfrak{q})=\mathfrak{o}}} \frac{\lambda_\varpi^{(\mathfrak{t})}(\mathfrak{m}\mathfrak{b})}{\sqrt{\mathcal{N}(\mathfrak{m})}} f(\mathfrak{m}\mathfrak{q}';s) \Bigg|.$$

Before we substitute this back into (118), we add a suitable positive contribution of the continuous spectrum, and with the notation (95) we majorize the characteristic function of

$$\big\{\nu\in\mathcal{S} : |1/4-\nu^2| \leqslant (\mathcal{N}\mathfrak{q})^\varepsilon\big\}$$

with $O_K(1)$ times the function

$$k(\nu) := \prod_{j=1}^d k_Z(\nu_j)\,, \quad Z := (\mathcal{N}\mathfrak{q})^{\varepsilon/2} \geqslant 1\,.$$

Using (119) and Lemma 3 we conclude that the ϖ-sum in (118) is

$$\ll_\varepsilon (\mathcal{N}\mathfrak{q})^{-1+2\theta+2\varepsilon} \max_{\mathfrak{b}_1,\mathfrak{b}_2|\mathfrak{c}_\pi} \sum_{\mathcal{N}\mathfrak{m}_1,\mathcal{N}\mathfrak{m}_2\ll LY/\mathcal{N}\mathfrak{q}} \big(\mathcal{N}(\mathfrak{m}_1\mathfrak{m}_2)\big)^{-1/2} \times \Bigg| \sum_{\varpi\in\mathcal{C}(\mathfrak{c})} \frac{1}{C_\varpi} \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}} k(\nu_\varpi)\bar{\lambda}_\varpi^{(\mathfrak{t})}(\mathfrak{m}_1\mathfrak{b}_1)\lambda_\varpi^{(\mathfrak{t})}(\mathfrak{m}_2\mathfrak{b}_2) + \mathrm{CSC}\Bigg| .$$

Note that $LY/\mathcal{N}\mathfrak{q} \ll_\varepsilon (\mathcal{N}\mathfrak{q})^\varepsilon L$. We are now in a position to apply the Kuznetsov formula (92) to the second line of the preceding display. We proceed very similarly as in the proof of Lemma 6 and estimate the right-hand side of (92) trivially, using (96) and (91). The diagonal contribution is

$$\ll_\varepsilon (\mathcal{N}\mathfrak{q})^{-1+2\theta+\varepsilon} \sum_{\mathcal{N}\mathfrak{m}\ll LY/\mathcal{N}\mathfrak{q}} (\mathcal{N}\mathfrak{m})^{-1}L^2 \ll (\mathcal{N}\mathfrak{q})^{-1+2\theta+2\varepsilon}L^2,$$

while the off-diagonal contribution is (use (97) with $\Xi \to (\mathcal{N}\mathfrak{q})^{d\varepsilon/2}$ and $\mathfrak{m} \to \gcd(\mathfrak{m}_1,\mathfrak{m}_2)$)

$$\begin{aligned} &\ll_\varepsilon (\mathcal{N}\mathfrak{q})^{-1+2\theta+\varepsilon} \sum_{\mathcal{N}\mathfrak{m}_1,\mathcal{N}\mathfrak{m}_2\ll LY/\mathcal{N}\mathfrak{q}} \big(\mathcal{N}\gcd(\mathfrak{m}_1,\mathfrak{m}_2,\mathfrak{c})\big)^{1/2}\big(\mathcal{N}(\mathfrak{m}_1\mathfrak{m}_2)\big)^{-1/4+\varepsilon} \\ &\ll_\varepsilon (\mathcal{N}\mathfrak{q})^{-1+2\theta+2\varepsilon}\Big(\sum_{\mathcal{N}\mathfrak{m}\ll LY/\mathcal{N}\mathfrak{q}} (\mathcal{N}\gcd(\mathfrak{m},\mathfrak{c}))^{1/4}(\mathcal{N}\mathfrak{m})^{-1/4}\Big)^2 \\ &\ll_\varepsilon (\mathcal{N}\mathfrak{q})^{-1+2\theta+3\varepsilon}L^{3/2}. \end{aligned}$$

Going back to (117) and using (118) it follows that the contribution of $\varpi \in \mathcal{C}(\mathfrak{c},\varepsilon)$ in (115) is

$$\ll_\varepsilon (\mathcal{N}\tilde{v})^{42d}(\mathcal{N}\mathfrak{q})^{-1/2+\theta+\varepsilon}L\,.$$

Together with (116) and the remarks preceding (117) this implies that (113) is at most

$$\ll_\varepsilon (\mathcal{N}\tilde{v})^{2B}(\mathcal{N}\mathfrak{q})^{-1/2+\theta+\varepsilon}L\,,$$

where $B = B(d,\varepsilon) > 0$ is a certain constant. By summing trivially over $\ell_{1,2}$ in the off-diagonal part of (111) and recalling also (112) we infer that

$$\mathcal{L}_{\chi_{\mathrm{fin}}}(v) \ll_\varepsilon (\mathcal{N}\tilde{v})^B(\mathcal{N}\mathfrak{q})^\varepsilon\big((\mathcal{N}\mathfrak{q})^{1/2}L^{-1/2} + (\mathcal{N}\mathfrak{q})^{1/4+\theta/2}L^{1/2}\big)\,.$$

The right-hand side is smallest when we take $L := (\mathcal{N}\mathfrak{q})^{1/4-\theta/2}$, then

$$\mathcal{L}_{\chi_{\mathrm{fin}}}(v) \ll_\varepsilon (\mathcal{N}\tilde{v})^B(\mathcal{N}\mathfrak{q})^{\frac{1}{2}-\frac{1}{8}(1-2\theta)+\varepsilon}.$$

This result in combination with (109) for $A := 2+B$ shows that (107) is at most

$$\ll_\varepsilon (\mathcal{N}\mathfrak{q})^{\frac{1}{2}-\frac{1}{8}(1-2\theta)+\varepsilon},$$

whence by (75)

$$L(1/2,\pi\otimes\chi) \ll_{\pi,\chi_\infty,\varepsilon} (\mathcal{N}\mathfrak{q})^{\frac{1}{2}-\frac{1}{8}(1-2\theta)+\varepsilon}.$$

The proof is complete.

3.4 Spectral decomposition of a Dirichlet series. We keep notation developed in sections 2 and 3.2. As another application of Theorem 2 we shall prove

Theorem 3. *Let π_1, π_2 be two irreducible cuspidal representations of $\mathrm{GL}_2(K)\backslash\mathrm{GL}_2(\mathbb{A})$ with the same unitary central character and signature character. Let $\ell_1, \ell_2 \in \mathfrak{o}$ be totally positive integers and write $\mathfrak{c} := \mathrm{lcm}(\ell_1\mathfrak{c}_{\pi_1}, \ell_2\mathfrak{c}_{\pi_2})$. Let $c, \beta \in \mathbb{N}_0$ such that $\beta > d(66+12c)$. Then for any $\varpi \in \mathcal{C}(\mathfrak{c}) \cup \mathcal{E}(\mathfrak{c})$ with $\varepsilon_\varpi = 1$ and for any $\mathfrak{t} \mid \mathfrak{c}\mathfrak{c}_\varpi^{-1}$ there exists a holomorphic function*

$$F_{\varpi,\mathfrak{t}} : \left\{ s \in \mathbb{C}^d \mid 1/2 + \theta < \Re s_j < 3/2 \right\} \to \mathbb{C}$$

depending only on $\pi_{1,2}$, β, ϖ, $\mathfrak{t}$, K such that the following two properties hold.

- *For an ideal $\mathfrak{y} \subseteq \mathfrak{o}$ and $0 << q \in \mathfrak{y}$ there is a spectral decomposition in the domain $1 < \Re s_j < 3/2$*

$$\sum_{\substack{\ell_1 r_1 - \ell_2 r_2 = q \\ 0 << r_{1,2} \in \mathfrak{y}}} \frac{\lambda_{\pi_1}(r_1\mathfrak{y}^{-1})\bar{\lambda}_{\pi_2}(r_2\mathfrak{y}^{-1}) N(\ell_1 r_1 \ell_2 r_2)^{(\beta-1)/2}}{\prod_{j=1}^d ((\ell_1 r_1 + \ell_2 r_2)^{\sigma_j})^{s_j+\beta-1}}$$
$$= \prod_{j=1}^d q_j^{1/2-s_j} \int_{(\mathfrak{c})} \sum_{\mathfrak{t} \mid \mathfrak{c}\mathfrak{c}_\varpi^{-1}} \lambda_\varpi^{(\mathfrak{t})}(q\mathfrak{y}^{-1}) F_{\varpi,\mathfrak{t}}(s) \, d\varpi \, .$$

- *For $0 < \varepsilon < 1/2$ there is a uniform bound in the domain $1/2+\theta+\varepsilon < \Re s_j < 3/2$*

$$\int_{(\mathfrak{c})} \sum_{\mathfrak{t} \mid \mathfrak{c}\mathfrak{c}_\varpi^{-1}} (\mathcal{N}\tilde{\lambda}_\varpi)^c \left| F_{\varpi,\mathfrak{t}}(s) \right| d\varpi \ll (\mathcal{N}\mathfrak{y})^{-1/2} \left(\mathcal{N}(\ell_1\ell_2) \right)^\varepsilon (\mathcal{N}\tilde{s})^{d(46+8c)}$$

  with the notation (108) and an implied constant depending only on $\pi_{1,2}, \beta, K, \varepsilon$. In particular, the left-hand side of the spectral identity can be continued holomorphically to the larger domain $\Re s_j > 1/2 + \theta$ with polynomial growth on vertical lines.

Remark 13. The character assumptions on $\pi_{1,2}$ and the positivity assumptions on $\ell_{1,2}$, $r_{1,2}$, q are not essential, they only serve simplicity of notation and exposition. With a little more work one can show that the implied constant depends polynomially on $C(\pi_1)C(\pi_2)$ and β.

Remark 14. Selberg [Se] asks for the meromorphic continuation of a Dirichlet series associated to shifted convolution sums. Progress over $\mathbb{Q}$ in this direction was made by Good [Go1,2], Sarnak [S1,2], Jutila [J1,2], Motohashi [Mo] and the authors [BlH2]. A version of Theorem 3 for $\pi_{1,2}$ whose archimedean components belong to the discrete series appears in [CoPS].

Proof. This is similar to the proof of Theorem 2 in [BlH2], so we present only the main steps, and omit convergence issues that are discussed in detail in [BlH2]. For c and β as in the statement, $\gamma := d(45+8c)$ and $t \geqslant 0$ we consider the functions

$$W_\beta(t; z) := t^{\beta/2} e^{-zt} \qquad \text{and} \qquad G_\gamma(t) := \begin{cases} (t(1-t))^\gamma \, , & 0 \leqslant t \leqslant 1 \, , \\ 0 \, , & t > 1 \, . \end{cases}$$

By Laplace inversion we have, for any $y_1, y_2, Y > 0$,

$$\left(\frac{y_1 y_2}{Y^2} \right)^{\beta/2} G_\gamma \left(\frac{y_1 + y_2}{Y} \right) = \frac{1}{2\pi i} \int_{(1)} \check{G}_\gamma(z) \, W_\beta \left(\frac{y_1}{Y}; z \right) W_\beta \left(\frac{y_2}{Y}; z \right) dz \, , \qquad (120)$$

where

$$\check{G}_\gamma(z) := \int_0^\infty G_\gamma(t) e^{zt} dt\,, \quad z \in \mathbb{C}\,.$$

We note the bound

$$\check{G}_\gamma(z) \ll_\gamma |z|^{-\gamma-1}, \quad \Re z = 1\,, \tag{121}$$

which follows easily by partial integration. We will also need the Mellin transform of $G_\gamma(t)$ as given by [GrR, 3.196.3],

$$\int_0^\infty G_\gamma(t)\, t^{s-1}\, dt = \frac{\Gamma(s+\gamma)\Gamma(\gamma+1)}{\Gamma(s+1+2\gamma)}\,, \quad \Re s > -\gamma\,. \tag{122}$$

Let $\Re z_j = 1$ and $Y_j > 0$ for $1 \leqslant j \leqslant d$ (we will later integrate over all z_j and Y_j), and let us write $z = (z_1, \dots z_d)$ and $Y = (Y_1, \dots Y_d)$. We apply Theorem 2 and Remark 12 with $a = 0$, $b = 1$, c as in the statement of Theorem 3, $W_1(y; z) := \prod_{j=1}^d W_\beta(y_j; z_j)$, $W_2(y; z) := \prod_{j=1}^d \bar{W}_\beta(y_j; z_j)$. We find

$$\sum_{\substack{\ell_1 r_1 - \ell_2 r_2 = q \\ 0 << r_{1,2} \in \mathfrak{y}}} \frac{\lambda_{\pi_1}(r_1\mathfrak{y}^{-1})\bar{\lambda}_{\pi_2}(r_2\mathfrak{y}^{-1})}{\sqrt{\mathcal{N}(r_1 r_2 \mathfrak{y}^{-2})}} \prod_{j=1}^d W_\beta\left(\frac{(\ell_1 r_1)^{\sigma_j}}{Y_j}; z_j\right) W_\beta\left(\frac{(\ell_2 r_2)^{\sigma_j}}{Y_j}; z_j\right)$$
$$= \int_{(\mathfrak{c})} \sum_{\mathfrak{t} | \mathfrak{c}\mathfrak{c}_\varpi^{-1}} \frac{\lambda_\varpi^{(\mathfrak{t})}(q\mathfrak{y}^{-1})}{\sqrt{\mathcal{N}(q\mathfrak{y}^{-1})}} W_{\varpi,\mathfrak{t}}\left(\frac{q^{\sigma_1}}{Y_1}, \dots, \frac{q^{\sigma_d}}{Y_d}; z\right) d\varpi \tag{123}$$

for some functions $W_{\varpi,\mathfrak{t}}(\cdot; z) : K_\infty^\times \to \mathbb{C}$ depending only on $\pi_{1,2}$, β, z, ϖ, $\mathfrak{t}$, K. Following Remark 12 and using $\|W_{1,2}(\cdot; z)\|_{A^\kappa} \ll (\mathcal{N}\tilde{z})^\kappa$ for $\beta > 3\kappa$ we see that

$$\int_{(\mathfrak{c})} \sum_{\mathfrak{t} | \mathfrak{c}\mathfrak{c}_\varpi^{-1}} (\mathcal{N}\tilde{\lambda}_\varpi)^c \big|W_{\varpi,\mathfrak{t}}(y; z)\big| d\varpi$$
$$\ll \big(\mathcal{N}(\ell_1\ell_2)\big)^{1/2+\varepsilon} (\mathcal{N}\tilde{z})^{d(44+8c)} \prod_{j=1}^d |y_j|^{1/2-\theta-\varepsilon} \min\big(1, |y_j|^{-1}\big)\,, \tag{124}$$

the implied constant depending only on $\pi_{1,2}$, β, c, K, ε. We integrate both sides of (123) against

$$\left(\frac{1}{2\pi i}\right)^d \int_{(1)} \cdots \int_{(1)} \prod_{j=1}^d \check{G}_\gamma(z_j) dz\,,$$

so that by (120)

$$\sum_{\substack{\ell_1 r_1 - \ell_2 r_2 = q \\ 0 << r_{1,2} \in \mathfrak{y}}} \lambda_{\pi_1}(r_1\mathfrak{y}^{-1})\bar{\lambda}_{\pi_2}(r_2\mathfrak{y}^{-1})\big(\mathcal{N}(\ell_1 r_1 \ell_2 r_2)\big)^{(\beta-1)/2} \prod_{j=1}^d Y_j^{-\beta} G_\gamma\left(\frac{(\ell_1 r_1 + \ell_2 r_2)^{\sigma_j}}{Y_j}\right)$$
$$= \big(\mathcal{N}(\ell_1\ell_2 q\mathfrak{y})\big)^{-1/2} \int_{(\mathfrak{c})} \sum_{\mathfrak{t} | \mathfrak{c}\mathfrak{c}_\varpi^{-1}} \lambda_\varpi^{(\mathfrak{t})}(q\mathfrak{y}^{-1}) H_{\varpi,\mathfrak{t}}\left(\frac{q^{\sigma_1}}{Y_1}, \dots, \frac{q^{\sigma_d}}{Y_d}\right) d\varpi\,, \tag{125}$$

where

$$H_{\varpi,\mathfrak{t}}(y) := \left(\frac{1}{2\pi i}\right)^d \int_{(1)} \cdots \int_{(1)} W_{\varpi,\mathfrak{t}}(y; z) \prod_{j=1}^d \check{G}_\gamma(z_j) dz\,.$$

Note that by (121) and (124) these integrals converge absolutely and they satisfy the bound

$$\int_{(\mathfrak{c})} \sum_{\mathfrak{t} \mid \mathfrak{c}\mathfrak{c}_{\varpi}^{-1}} (\mathcal{N}\tilde{\lambda}_{\varpi})^{c} \big|H_{\varpi,\mathfrak{t}}(y)\big| d\varpi \ll_{\pi_{1,2},\beta,c,K,\varepsilon} \big(\mathcal{N}(\ell_1\ell_2)\big)^{1/2+\varepsilon} \prod_{j=1}^{d} |y_j|^{1/2-\theta-\varepsilon} \min\big(1, |y_j|^{-1}\big) . \quad (126)$$

Now for $s \in \mathbb{C}^d$ with $1 < \Re s_j < 3/2$ we integrate both sides of (125) against

$$\int_{K_{\infty,+}^{\times}} \prod_{j=1}^{d} Y_j^{1-s_j} \, d^{\times} Y$$

and use also (122): we arrive at the spectral identity of Theorem 3 with

$$F_{\varpi,\mathfrak{t}}(s) := \big(\mathcal{N}(\ell_1\ell_2\mathfrak{y})\big)^{-1/2} \left(\prod_{j=1}^{d} \frac{\Gamma(s_j+\beta+2\gamma)}{\Gamma(s_j+\beta-1+\gamma)\Gamma(\gamma+1)} \right) \int_{K_{\infty,+}^{\times}} H_{\varpi,\mathfrak{t}}(y) \prod_{j=1}^{d} y_j^{s_j-1} d^{\times} y \, .$$

By (126) these functions are holomorphic in the domain $1/2+\theta+\varepsilon < \Re s_j < 3/2$ and there they satisfy the bound of Theorem 3. The proof is complete. □

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Received: April 16, 2009
Revision: May 5, 2009
Accepted: October 7, 2009