# ERRATUM TO "TWISTED $L$-FUNCTIONS OVER NUMBER FIELDS AND HILBERT'S ELEVENTH PROBLEM"

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**1.** The sentence on lines -3 to -2 of [BH, p. 7] should read as follows: The bundle $H$ is trivial, because any $\varphi(0) \in H(0)$ extends to a section $\varphi \in H$ satisfying $\varphi(s,g) = \varphi(0,g)H(g)^s$, where $H(g)$ is the height function defined before [GJ, (3.3)].

**2.** On line 3 of [BH, p. 8], the right hand side should read, in accordance with [GJ, (3.15)],

$$\frac{2}{\pi}\int_0^\infty \int_{K^\times\backslash\mathbb{A}^1}\int_{\mathcal{K}} \varphi_1\left(iy, \begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} k\right) \bar{\varphi}_2\left(iy, \begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} k\right) dk\, da\, dy.$$

**3.** Lines -11 to -9 of [BH, p. 11] should read as follows: By [BrMo, §4], the functions $\tilde{W}_{q/2,\nu}$ ($q \in \mathbb{Z}$) for fixed $\nu$ and fixed parity $\kappa \in \{0,1\}$ form an orthonormal basis of the Hilbert space $L^2(\mathbb{R}^\times, d^\times y)$ which justifies our normalization:

$$L^2(\mathbb{R}^\times, d^\times y) = \bigoplus_{q \equiv \kappa \,(\mathrm{mod}\ 2)} \mathbb{C}\tilde{W}_{\frac{q}{2},\nu}, \qquad \langle \tilde{W}_{\frac{q}{2},\nu}, \tilde{W}_{\frac{q'}{2},\nu}\rangle = \delta_{q,q'}. \tag{25}$$

**4.** On line -7 of [BH, p. 30], we stated incorrectly that any element $g \in \mathrm{GL}_2(\mathbb{A})$ can be written as

$$g = z\tilde{\gamma}\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}(\tilde{k}_\infty \times \tilde{k}_{\mathrm{fin}})$$

for some $z \in Z(\mathbb{A})$, $\tilde{\gamma} \in \mathrm{GL}_2(K)$, $\tilde{k}_\infty \times \tilde{k}_{\mathrm{fin}} \in \mathrm{SO}_2(K_\infty) \times \mathcal{K}(\mathfrak{c}_\pi)$, and $\left(\begin{smallmatrix} y & x \\ 0 & 1\end{smallmatrix}\right) \in P(\mathbb{A})$, where $y = y_\infty \times y_{\mathrm{fin}}$ is such that all coordinates of $y_\infty$ exceed $\delta$ and $y_{\mathrm{fin}}$ takes values from a finite set depending only on $K$ and $\mathfrak{c}_\pi$. Instead, we can only deduce that

$$g = z\tilde{\gamma}\left(\begin{pmatrix} y' & x' \\ 0 & 1\end{pmatrix} \times h\right)(\tilde{k}_\infty \times \tilde{k}_{\mathrm{fin}}),$$

where $\left(\begin{smallmatrix} y' & x' \\ 0 & 1\end{smallmatrix}\right) \in P(K_\infty)$ with $y'_1, \dots, y'_d > \delta$, and $h \in \mathrm{GL}_2(\mathbb{A}_{\mathrm{fin}})$ takes values from a finite set depending only on $K$ and $\mathfrak{c}_\pi$. That is, our mistake was to assume that the matrices $h$ are upper triangular.

As we shall explain below, the weaker statement suffices for the proof of Lemma 5 in [BH]. More precisely, we shall show that if $g$ is decomposed as above and $\phi \in V_{\pi,q}(\mathfrak{c}_\pi)$ is arbitrary, then

$$|\phi(g)| \ll_{\pi,K} \|\phi\| \sum_{\substack{r \in R \\ r \neq 0}} |\tilde{W}_{q/2,\nu_\pi}(ry')|,$$

where $R \in I(K)$ is a fractional ideal depending only on $K$ and $\mathfrak{c}_\pi$. From here the argument can be finished as on [BH, p. 31], with the only change that $y_\infty$ and $(y_{\mathrm{fin}}^{-1})$ are replaced by $y'$ and $R$.

If $g$ is decomposed as above, then

$$\phi(g) = \psi\left(\begin{pmatrix} y' & x' \\ 0 & 1\end{pmatrix}\right),$$

where $\psi$ denotes the right $h$-translate of $\phi$. We shall regard this as a value of the Hilbert modular form

$$(x_1+iy_1,\dots,x_d+iy_d)\mapsto\psi\left(\begin{pmatrix}y & x\\ 0 & 1\end{pmatrix}\right),\qquad y\in K_\infty^\times,\quad x\in K_\infty.$$

Analogously to [BH, (30)], there is a Fourier decomposition

$$\psi\left(\begin{pmatrix}y & x\\ 0 & 1\end{pmatrix}\right)=\sum_{\substack{r\in R\\ r\neq 0}}\rho_\psi(r)\tilde{W}_{q/2,\nu_\pi}(ry)e\left(r^{\sigma_1}x_1+\dots+r^{\sigma_d}x_d\right),\qquad y\in K_\infty^\times,\quad x\in K_\infty,$$

where $R\in I(K)$ is a fractional ideal depending only on $K$ and $\mathfrak{c}_\pi$. By the normalization of the Whittaker function, the coefficients $\rho_\psi(r)$ remain unchanged if $\psi$ is replaced by any of its nonzero Maaß shifts. In particular, if $\tilde\psi$ denotes the nonzero Maaß shift of $\psi$ of minimal weight $\tilde q\in\mathbb{Z}^d$, then

$$\tilde\psi\left(\begin{pmatrix}y & x\\ 0 & 1\end{pmatrix}\right)=\sum_{\substack{r\in R\\ r\neq 0}}\rho_\psi(r)\tilde{W}_{\tilde q/2,\nu_\pi}(ry)e\left(r^{\sigma_1}x_1+\dots+r^{\sigma_d}x_d\right),\qquad y\in K_\infty^\times,\quad x\in K_\infty,$$

where the function $\tilde{W}_{\tilde q/2,\nu_\pi}$ now depends only on $\pi$ and $K$. We shall use this observation to prove the uniform bound $\rho_\psi(r)\ll_{\pi,K}\|\phi\|$, which then implies our claim above:

$$|\phi(g)|=\left|\psi\left(\begin{pmatrix}y' & x'\\ 0 & 1\end{pmatrix}\right)\right|\leqslant\sum_{\substack{r\in R\\ r\neq 0}}|\rho_\psi(r)\tilde{W}_{q/2,\nu_\pi}(ry')|\ll_{\pi,K}\|\phi\|\sum_{\substack{r\in R\\ r\neq 0}}|\tilde{W}_{q/2,\nu_\pi}(ry')|.$$

Our starting point is the Plancherel identity

$$\sum_{\substack{r\in R\\ r\neq 0}}|\rho_\psi(r)\tilde{W}_{\tilde q/2,\nu_\pi}(ry)|^2=\int_{K_\infty/R^0}\left|\tilde\psi\left(\begin{pmatrix}y & x\\ 0 & 1\end{pmatrix}\right)\right|^2dx,$$

where $R^0$ denotes the dual lattice of $R\subset K_\infty$. We keep a single $r\in R$ on the left hand side, and then integrate both sides over

$$\mathcal{F}(r):=\left\{y\in K_\infty^\times\mid |y_j|>1/|r^{\sigma_j}|\right\},$$

with respect to the measure $dy/y^2$. We obtain

$$|\rho_\psi(r)|^2\,|\mathcal{N}r|\ll_{\pi,K}\int_{(K_\infty/R^0)\times\mathcal{F}(r)}\left|\tilde\psi\left(\begin{pmatrix}y & x\\ 0 & 1\end{pmatrix}\right)\right|^2\frac{dx\,dy}{y^2}.$$

By a standard argument (cf. [Iw, Lemma 2.10]), the Siegel set $(K_\infty/R^0)\times\mathcal{F}(r)$ covers each point in a fixed fundamental domain for $\tilde\psi$ with multiplicity $\ll_{\pi,K}|\mathcal{N}r|$, hence

$$|\rho_\psi(r)|^2\ll_{\pi,K}\|\tilde\psi\|^2=\|\psi\|^2=\|\phi\|^2.$$

The desired bound $\rho_\psi(r)\ll_{\pi,K}\|\phi\|$ follows.

**5.** In lines -5 to -1 of [BH, p. 32], the ideal classes should be understood in the narrow sense, while the generator $\gamma$ and the product $r_1r_2$ should be totally positive. Along with this change, the Kuznetsov formula [BH, (92)] should be corrected as follows: on the left hand side the restriction $\varepsilon_\pi=1$ should be omitted, and on the right hand side the summation over $U/U^2$ should be restricted to $U^+/U^2$. A detailed proof of the corrected formula appears in [Ma1] for a wide class of test functions including the ones we need [BH, (95)]. The proof is similar to what we outlined on [BH, p. 33–35], but the analysis is carried out on the larger space[1]

$$\mathrm{FS} := L^2(\mathrm{GL}_2(K)Z(K_\infty)\backslash \mathrm{GL}_2(\mathbb{A})/\mathcal{K}(c)) = \bigoplus_{\omega\in\widehat{C(K)}} L^2(\mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbb{A})/\mathcal{K}(\mathfrak{c}),\omega).$$

In particular, whenever we refer to $L^2(\mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbb{A})/T\mathcal{K}(\mathfrak{c}),\omega)$ in [BH], it should be understood as $L^2(\mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbb{A})/\mathcal{K}(\mathfrak{c}),\omega)$. Accordingly, each restriction $\varepsilon_\pi = 1$ or $\varepsilon_\varpi = 1$ should be disregarded in the text, e.g. the notation preceding [BH, Theorem 2] should read

$$\int_{(\mathfrak{c})} f_\varpi \, d\varpi := \sum_{\pi\in\mathcal{C}(\mathfrak{c})} f_\pi + \int_{\varpi\in\mathcal{E}(\mathfrak{c})} f_\varpi \, d\varpi.$$

Then [BH, Lemma 6] and [BH, Theorems 2–3] remain valid, and for the latter we do not need to assume that $\pi_1$ and $\pi_2$ have the same signature character, cf. [BH, Remarks 11 & 13].

**6.** In lines -10 to -9 of [BH, p. 45], all five occurrences of $\mathfrak{c}$ should be $\mathfrak{t}$, see [Ma2] for a detailed proof.

## References


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[1] In retrospect, our mistake was to treat the group $\mathrm{O}_2(K_\infty)$ as if it were commutative, leading us to the false belief that the finite subgroup $T$ acts by scalars on any $\pi\in\mathcal{C}(\mathfrak{c})$ and $\varpi\in\mathcal{E}(\mathfrak{c})$.