HYBRID BOUNDS FOR TWISTED L-FUNCTIONS

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Abstract. The aim of this paper is to derive bounds on the critical line $\Re s = \frac{1}{2}$ for L-functions attached to twists $f \otimes \chi$ of a primitive cusp form f of level N and a primitive character modulo q that break convexity simultaneously in the s and q aspects. If f has trivial nebentypus, it is shown that

$$L(f \otimes \chi, s) \ll (N|s|q)^{\varepsilon} N^{\frac{4}{5}} (|s|q)^{\frac{1}{2}-\frac{1}{40}},$$

where the implied constant depends only on $\varepsilon > 0$ and the archimedean parameter of f. To this end, two independent methods are employed to show

$$L(f \otimes \chi, s) \ll (N|s|q)^{\varepsilon} N^{\frac{1}{2}} |s|^{\frac{1}{2}} q^{\frac{3}{8}}$$

and

$$L(g, s) \ll D^{\frac{2}{3}} |s|^{\frac{5}{12}}$$

for any primitive cusp form g of level D and arbitrary nebentypus (not necessarily a twist $f \otimes \chi$ of level $D \mid Nq^2$).

1. Introduction

In the past two decades, powerful methods have been obtained to study the growth of L-functions on the critical line $\Re s = \frac{1}{2}$. Depending on the application, one usually tries to break the convexity bound in one of the parameters of the L-function while keeping the dependence of the other parameters polynomial. By now there are only two results where subconvexity in two parameters has been achieved simultaneously: Heath-Brown [HB] combined Burgess' and van der Corput's method to obtain

$$L(\chi, s) \ll_{\varepsilon} (|s|q)^{\frac{3}{16}+\varepsilon} \tag{1.1}$$

for Dirichlet L-functions for a character χ modulo q on the line $\Re s = \frac{1}{2}$. Very recently, Jutila and Motohashi [JM] managed to obtain uniform subconvexity in the archimedean and the s-aspect for L-functions for cusp forms on GL_2. They showed

$$L(f, s) \ll_{\varepsilon} (|s| + |\mu|)^{\frac{1}{3}+\varepsilon}$$

for any holomorphic or non-holomorphic cusp form f for the full modular group where we write

$$\mu = t_f := \begin{cases} \sqrt{\lambda - \frac{1}{4}} & \text{when } f \text{ is a Maaß form of Laplacian eigenvalue } \lambda, \\ \frac{(1-k)i}{2} & \text{when } f \text{ is a holomorphic form of weight } k, \end{cases} \tag{1.2}$$

and refer to μ as the archimedean parameter of f. This was a major breakthrough, and the proof is long and very elaborate. For most arithmetic applications, however, the focus lies on the non-archimedean parameter ("conductor") of the L-function. For L-functions attached to general cusp forms for a congruence subgroup $\Gamma_0(q)$, the authors and Philippe Michel [BHM2] recently obtained

$$L(f, s) \ll_{s,\mu} q^{\frac{1}{4}-\frac{1}{1889}},$$

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but by present technology it seems to be out of reach to break simultaneously the convexity bound in q and one of the other parameters. We can, however, get hybrid bounds, if we restrict ourselves to a special subfamily of cusp forms, namely those that occur as a twist of a fixed form. In analogy with (1.1), we shall prove the following hybrid estimate.

Theorem 1. *Let f be a primitive (holomorphic or Maaß) cusp form of archimedean parameter μ as in (1.2), level N and trivial nebentypus, and let χ be a primitive character modulo q. Then for $\Re s = \frac{1}{2}$ and for any $\varepsilon > 0$ the twisted L-function $L(f \otimes \chi, s)$ satisfies*

$$L(f \otimes \chi, s) \ll_{\mu,\varepsilon} (N|s|q)^{\varepsilon} N^{\frac{4}{5}} (|s|q)^{\frac{1}{2}-\frac{1}{40}},$$

where the implied constant depends only on ε and μ.

Note that $f \otimes \chi$ is a primitive cusp form of level dividing Nq^2 and nebentypus χ^2. A thorough examination of the proof shows that the dependence on the archimedean parameter μ of f can be made polynomial. In order to prove Theorem 1, we combine two methods each of which gives subconvexity in only one of the parameters. Theorem 1 will be a simple corollary from Theorems 2 and 3 below. Pushing a method of Bykovskiĭ [By] to its limit, we shall show

Theorem 2. *Let f be a primitive (holomorphic or Maaß) cusp form of archimedean parameter μ, level N and trivial nebentypus, and let χ be a primitive character modulo q. Then for $\Re s = \frac{1}{2}$ and for any $\varepsilon > 0$ the twisted L-function satisfies*

$$L(f \otimes \chi, s) \ll_{\varepsilon} \left(|s|^{\frac{1}{4}}|\mu|^{\frac{1}{2}}N^{\frac{1}{4}}q^{\frac{3}{8}} + |s|^{\frac{1}{2}}|\mu|N^{\frac{1}{2}}(N,q)^{\frac{1}{4}}q^{\frac{1}{4}}\right)(|s||\mu|Nq)^{\varepsilon}$$

if f is holomorphic, and

$$L(f \otimes \chi, s) \ll_{\varepsilon} \left(|s|^{\frac{1}{4}}(1+|\mu|)^3N^{\frac{1}{4}}q^{\frac{3}{8}} + |s|^{\frac{1}{2}}(1+|\mu|)^{\frac{7}{2}}N^{\frac{1}{2}}(N,q)^{\frac{1}{4}}q^{\frac{1}{4}}\right)(|s|(1+|\mu|)Nq)^{\varepsilon}$$

otherwise.

In [BHM1] the authors obtained

$$L(f \otimes \chi, s) \ll_{\varepsilon} (|s|(1+|\mu|)Nq)^{\varepsilon}|s|^{\alpha}(1+|\mu|)^{\beta}N^{\gamma}q^{\frac{1}{2}-\delta}$$

with $\alpha = \frac{503}{256}$, $\beta = \frac{1285}{256}$, $\gamma = \frac{13}{16}$, $\delta = \frac{25}{256}$ in the more general setting where f was allowed to have any nebentypus. Theorem 2 is now a complete analogue of Burgess' result [Bu] for Dirichlet L-functions in the q-aspect; note that it is—unlike its predecessor in [BHM1]—independent of the Ramanujan–Petersson conjecture. As in [BHM1], Theorem 2 can also be used as an input for certain automorphic functions on GL_4. Together with the convexity bound, we obtain

$$L(f \otimes \chi, s) \ll_{\varepsilon} (|s|(1+|\mu|)Nq)^{\varepsilon}|s|^{\frac{1}{2}}(1+|\mu|)^3N^{\frac{1}{2}}q^{\frac{3}{8}} \tag{1.3}$$

from Theorem 2. Changing exponents in [HM], we obtain[1]

Corollary 1. *Let f and g be two primitive (holomorphic or Maaß) cusp forms of respective levels q, D and respective nebentypus χ_f, χ_g such that $\chi_f\chi_g$ is non-trivial. Then for $\Re s = \frac{1}{2}$ the associated Rankin–Selberg L-function satisfies*

$$L(f \otimes g, s) \ll ((|s| + |t_f| + |t_g|)D)^A q^{\frac{1}{2}-\frac{1}{1413}},$$

where $A > 0$ is an absolute constant.

Waldspurger's theorem translates bounds for twisted modular L-functions into bounds for the coefficients of half-integral weight modular forms. Theorem 2 gives

[1] Mathematica source code available upon request.

Corollary 2. *Let $k, M \in \mathbb{N}$, and let χ be a character modulo $4M$. Let*

$$f(z) := \sum_{n=1}^{\infty} \rho_f(n)(4\pi n)^{\frac{k}{2}+\frac{1}{4}} e(nz)$$

be an L^2-normalized cusp form in $S'_{k+\frac{1}{2}}(4M, \chi)$, where $S'_{k+\frac{1}{2}}(4M, \chi)$ denotes the orthogonal complement in $S_{k+\frac{1}{2}}(4M, \chi)$ of the space of theta series in one variable[2]. Then for any $\varepsilon > 0$ and any $n \geqslant 1$ we have

$$\sqrt{n}\rho_f(n) \ll_\varepsilon (kMn)^\varepsilon \left(\Gamma\left(k+\frac{1}{2}\right)\right)^{-1/2} \left(k^{\frac{3}{4}} M^{\frac{3}{4}} n^{\frac{3}{16}} + kMn^{\frac{1}{8}}\right) (n, (2M)^\infty)^{\frac{5}{8}}.$$

The first nontrivial bound for Fourier coefficients of half-integral weight was proved by Iwaniec [Iw1]: $\sqrt{n}\rho_f(n) \ll_{k,\varepsilon} n^{3/14+\varepsilon}$ uniformly in M. For $k+\frac{1}{2} \geqslant \frac{5}{2}$, Bykovskiĭ [Bu] obtained $\sqrt{n}\rho_f(n) \ll_{M,k,\varepsilon} n^{3/16+\varepsilon}$ with an unspecified dependence on k and M. Various applications of Corollary 2 to ternary quadratic forms can be found in [Bl2]. The proof is based on a careful calculation of the Kohnen–Zagier constant appearing in Waldspurger's theorem. This has been carried out by Zhengyu Mao in [BHM1, Appendix 2] for every index n whose square part is coprime with $2M$. In Section 9 we give some refinements of the argument in [BHM1, Appendix 2] and indicate how to cover all indices n, as kindly communicated to us by Zhengyu Mao.

Since Theorem 2 is on the edge with respect to s, we obtain a version of Theorem 1 as soon as we have subconvexity in s with polynomial growth in q. There are several methods to break convexity in the s-aspect, but all of these have only been carried out for cusp forms for the full modular group. Although it is clearly known to experts in the field that a result of this kind for congruence subgroups can be achieved, the generalization is not completely straightforward. Probably the most elementary approach is due to Jutila [Ju1, Me], using only Voronoi summation and estimates for certain exponential integrals. It turns out, however, that this method is not directly applicable for congruence subgroups, since Voronoi summation is only available for certain fractions, and it is not clear what approximation properties Farey fractions with congruence restrictions have. Chronologically the first to obtain subconvexity (for holomorphic cusp forms of full level) in the s-aspect was Good [Go1, Go2] who deduced it from an asymptotic formula of the kind

$$\int_0^T \left|L\left(g, \frac{1}{2}+it\right)\right|^2 dt = c_1 T \log T + c_2 T + O(T^{1-\delta}). \tag{1.4}$$

Evaluating this integral leads to a shifted convolution problem in the coefficients $\lambda(n)$ of $L(g,s)$. There are several ways to obtain good bounds for such sums. Good [Go1, Go2] and many others (see, for example, [Ju2, Ju3, JM, Sa, LLY]) used a spectral decomposition for the Dirichlet series $\sum \lambda(n)\lambda(n+h)(n+h)^{-s}$. This approach has certain difficulties[3] in the non-holomorphic case (see e.g. [Sa]), but it can be made work since the shifting parameter is not too large. However, we prefer to use a more elementary method based on Jutila's variant of the circle method and Kuznetsov's trace formula that is readily available in the general case and does not require much calculation. Theorem 2 in [Bl1] yields

Proposition 1. *Let g be a primitive (holomorphic or weight zero Maaß) cusp form of archimedean parameter μ, level D and arbitrary nebentypus. Let $\theta \leqslant \frac{7}{64}$ be as in (2.4).*

a) Let K, T be large parameters such that

$$D \leqslant K^{\frac{\theta}{2}} T^{-\frac{\theta}{4}}, \qquad D^{\frac{9}{10-8\theta}} T^{\frac{4-2\theta}{5-4\theta}} \leqslant K < T,$$

or

$$K^{\frac{\theta}{2}} T^{-\frac{\theta}{4}} \leqslant D \leqslant T^{\frac{1}{5}}, \qquad DT^{\frac{4}{5}} \leqslant K < T.$$

[2]Note that $S'_{k+\frac{1}{2}}(4M, \chi)$ is the entire space $S_{k+\frac{1}{2}}(4M, \chi)$ if $k \geqslant 2$, while for $k = 1$ it equals the subspace $V(4M; \chi)$ defined in [U93].

[3]Very recently the authors found an alternate spectral decomposition that avoids these difficulties, see [BH].

Then the L-function attached to g satisfies

$$\int_T^{T+K}\left|L\left(g,\frac{1}{2}+it\right)\right|^2 dt \ll_{\mu,\varepsilon} (DT)^\varepsilon K.$$

b) If $D \geqslant T^{1/5}$ *then*

$$\int_0^T \left|L\left(g,\frac{1}{2}+it\right)\right|^2 dt \ll_{\mu,\varepsilon} (DT)^\varepsilon \min\left(D^{\frac{5}{2}}T^{\frac{1}{2}}, D^{\frac{1}{2}}T\right).$$

A more careful reasoning would give an asymptotic formula as in (1.4). As a simple consequence we obtain

Theorem 3. *Let g be a primitive (holomorphic or weight zero Maaß) cusp form of archimedean parameter μ, level D and arbitrary nebentypus. Let* $\theta \leqslant \frac{7}{64}$ *be as in* (2.4)*. For* $\Re s = \frac{1}{2}$ *we have*

$$L(g,s) \ll_{\mu,\varepsilon} (D|s|)^\varepsilon D^{\frac{13}{20}}|s|^{\frac{2-\theta}{5-4\theta}} \ll D^{\frac{2}{3}}|s|^{\frac{5}{12}}.$$

Precisely,

$$L(g,s) \ll_{\mu,\varepsilon} (D|s|)^\varepsilon \times \begin{cases} D^{\frac{9}{20-16\theta}}|s|^{\frac{2-\theta}{5-4\theta}}, & \text{if} \quad D \leqslant |s|^{\frac{3\theta}{20-25\theta}}; \\ D^{\frac{1}{2}}|s|^{\frac{2}{5}}, & \text{if} \quad |s|^{\frac{3\theta}{20-25\theta}} \leqslant D \leqslant |s|^{\frac{1}{5}}; \\ D^{\frac{5}{4}}|s|^{\frac{1}{4}}, & \text{if} \quad |s|^{\frac{1}{5}} \leqslant D. \end{cases} \tag{1.5}$$

Remark. Inequality (1.5) breaks the convexity bound in the s-aspect as long as $D \leqslant |s|^{\frac{1}{4}}$.

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2. Overview: automorphic forms

In this section we briefly compile some results from the theory of automorphic forms and introduce the relevant notation.

2.1. **Hecke eigenbases.** Let $D \geqslant 1$ be an integer, ψ be an even character to modulus D; let $k \geqslant 2$ be an even integer. We denote by $\mathcal{S}_k(D,\psi)$, $\mathcal{L}^2(D,\psi)$ and $\mathcal{L}^2_0(D,\psi) \subset \mathcal{L}^2(D,\psi)$, respectively, the Hilbert spaces (with respect to the Petersson inner product) of holomorphic cusp forms of weight k, of Maaß forms of weight zero, and of Maaß cusp forms of weight zero, with respect to the congruence subgroup $\Gamma_0(D)$ and with nebentypus ψ. These spaces are endowed with the action of the (commutative) algebra $\mathbf{T}$ generated by the Hecke operators $\{T_n \mid n \geqslant 1\}$. Moreover, the subalgebra $\mathbf{T}^{(D)}$ generated by $\{T_n \mid (n,D)=1\}$ is made of normal operators. As an immediate consequence, the spaces $\mathcal{S}_k(D,\psi)$ and $\mathcal{L}^2_0(D,\psi)$ have an orthonormal basis made of eigenforms of $\mathbf{T}^{(D)}$ and such a basis can be chosen to contain all L^2-normalized Hecke eigen-newforms (in the sense of Atkin–Lehner theory). We denote these bases by $\mathcal{B}_k(D,\psi)$ and $\mathcal{B}(D,\psi)$ respectively. For the rest of this paper we assume that any such basis satisfies these properties.

The orthogonal complement to $\mathcal{L}^2_0(D,\psi)$ in $\mathcal{L}^2(D,\psi)$ is the Eisenstein spectrum $\mathcal{E}(D,\psi)$ (plus possibly the space of constant functions if ψ is trivial). The adelic reformulation of the theory of modular forms provides a natural spectral expansion of this space in which the basis of Eisenstein series is indexed by a set of parameters of the form[4]

$$\{(\psi_1,\psi_2,f) \mid \psi_1\psi_2=\psi,\ f \in \mathcal{B}(\psi_1,\psi_2)\}, \tag{2.1}$$

where (ψ_1,ψ_2) ranges over the pairs of characters of modulus D such that $\psi_1\psi_2=\psi$ and $\mathcal{B}(\psi_1,\psi_2)$ is some finite set depending on (ψ_1,ψ_2) (specifically, $\mathcal{B}(\psi_1,\psi_2)$ corresponds to an orthonormal basis in the space of an induced representation constructed out of the pair (ψ_1,ψ_2), but we need not be

[4]We suppress here the independent spectral parameters $\frac{1}{2}+it$ with $t \in \mathbb{R}$.

more precise). We refer to [GJ] for the definition of these parameters as well as for the proof of the spectral expansion of the following form: for $g \in \mathcal{E}(D,\psi)$ one has

$$g(z) = \sum_{\substack{\psi_1\psi_2=\psi \\ f\in\mathcal{B}(\psi_1,\psi_2)}}\sum \int_{\mathbb{R}} \langle g, E_{\psi_1,\psi_2,f,t}\rangle E_{\psi_1,\psi_2,f,t}(z)\frac{dt}{4\pi}.$$

An important feature of this basis is that it consists of Hecke eigenforms for $\mathbf{T}^{(D)}$: for $(n,D)=1$ one has

$$T_n E_{\psi_1,\psi_2,f,t}(z) = \lambda_{\psi_1,\psi_2,t}(n)E_{\psi_1,\psi_2,f,t}(z)$$

with

$$\lambda_{\psi_1,\psi_2,t}(n) = \sum_{ab=n}\psi_1(a)a^{it}\psi_2(b)b^{-it}.$$

2.2. **Hecke eigenvalues and Fourier coefficients.** Let f be any such Hecke eigenform and let $\lambda_f(n)$ denote the corresponding eigenvalue for T_n; then for $(mn,D)=1$ one has

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)}\psi(d)\lambda_f(mn/d^2),$$

$$\overline{\lambda_f(n)} = \overline{\psi(n)}\lambda_f(n).$$

In particular, for $(mn,D)=1$ it follows that

$$\lambda_f(m)\overline{\lambda_f(n)} = \overline{\psi(n)}\sum_{d|(m,n)}\psi(d)\lambda_f(mn/d^2). \tag{2.2}$$

By [DFI, Proposition 19.6] we have

$$\sum_{n\leqslant x}|\lambda_f(n)|^2 \ll_\varepsilon ((1+|t_f|)Dx)^\varepsilon x \tag{2.3}$$

for any $x\geqslant 1$, $\varepsilon>0$.

We write the Fourier expansion of a modular form f as follows ($z=x+iy$):

$$f(z) = \sum_{n\geqslant 1}\rho_f(n)n^{k/2}e(nz) \quad \text{for} \quad f\in\mathcal{B}_k(D,\psi),$$

$$f(z) = \sum_{n\neq 0}\rho_f(n)W_{0,it_f}(4\pi|n|y)e(nx) \quad \text{for} \quad f\in\mathcal{B}(D,\psi),$$

and for a basis Eisenstein series

$$E_{\psi_1,\psi_2}(z,f,\tfrac{1}{2}+it) = c_{1,f,t}y^{1/2+it} + c_{2,f,t}y^{1/2-it} + \sum_{n\neq 0}\rho_{f,t}(n)W_{0,it}(4\pi|n|y)e(nx).$$

Here t_f denotes the spectral parameter (1.2) for which the currently best approximation is due to Kim–Sarnak [KS][5]:

$$|\Im t_f| \leqslant \theta := \frac{7}{64}. \tag{2.4}$$

When f is a Hecke eigenform, there is a close relationship between the Fourier coefficients of f and its Hecke eigenvalues $\lambda_f(n)$: one has for $(m,D)=1$ and any $n\geqslant 1$,

$$\lambda_f(m)\sqrt{n}\rho_f(n) = \sum_{d|(m,n)}\psi(d)\sqrt{\frac{mn}{d^2}}\rho_f\left(\frac{mn}{d^2}\right); \tag{2.5}$$

in particular, for $(m,D)=1$,

$$\lambda_f(m)\rho_f(1) = \sqrt{m}\rho_f(m). \tag{2.6}$$

[5]For Maaß cusp forms f of weight 1, $t_f \in \mathbb{R}$.

Moreover, these relations hold for all m, n if f is a newform.

We will also need the following lower bounds for any L^2-normalized newform f in either $\mathcal{B}_k(D,\psi)$ or $\mathcal{B}(D,\psi)$:

$$|\rho_f(1)|^2 \gg_\varepsilon \begin{cases} (4\pi)^{k-1}((k-1)!D)^{-1}(kD)^{-\varepsilon}, & \text{for} \quad f \in \mathcal{B}_k(D,\psi), \\ \cosh(\pi t_f)D^{-1}(D+|t_f|)^{-\varepsilon}, & \text{for} \quad f \in \mathcal{B}(D,\psi), \end{cases} \tag{2.7}$$

cf. [DFI, (6.22)–(6.23), (7.15)–(7.16)] and [HM, (31)].

2.3. **The trace formula.** Let $\phi : [0,\infty) \to \mathbb{C}$ be a smooth function satisfying $\phi(0) = \phi'(0) = 0$, $\phi^{(j)}(x) \ll_\varepsilon (1+x)^{-2-\varepsilon}$ for $0 \leqslant j \leqslant 3$. Let

$$\begin{aligned} \dot{\phi}(k) &:= i^k \int_0^\infty J_{k-1}(x)\phi(x)\frac{dx}{x}, \\ \tilde{\phi}(t) &:= \frac{i}{2\sinh(\pi t)} \int_0^\infty \left(J_{2it}(x) - J_{-2it}(x)\right)\phi(x)\frac{dx}{x} \end{aligned} \tag{2.8}$$

be Bessel transforms. Then for positive integers m, n the trace formula of Bruggeman–Kuznetsov holds:

$$\begin{aligned} \sum_{D\mid c} \frac{1}{c} S_\psi(m,n,c)\phi\left(\frac{4\pi\sqrt{mn}}{c}\right) &= \sum_{\substack{k\geqslant 2 \text{ even} \\ f\in\mathcal{B}_k(D,\psi)}}\sum \dot{\phi}(k)\frac{(k-1)!\sqrt{mn}}{\pi(4\pi)^{k-1}}\overline{\rho_f(m)}\rho_f(n) \\ + \sum_{f\in\mathcal{B}(D,\psi)} \tilde{\phi}(t_f)\frac{4\pi\sqrt{mn}}{\cosh(\pi t_f)}\overline{\rho_f(m)}\rho_f(n) &+ \sum_{\substack{\psi_1\psi_2=\psi \\ f\in\mathcal{B}(\psi_1,\psi_2)}}\sum \int_{-\infty}^{\infty} \tilde{\phi}(t)\frac{\sqrt{mn}}{\cosh(\pi t)}\overline{\rho_{f,t}(m)}\rho_{f,t}(n)\,dt, \end{aligned} \tag{2.9}$$

where the right-hand side runs over the spectrum of the Laplacian of weight zero in (2.9) acting on forms of level D and character ψ (cf. [Iw2, Theorems 9.4 and 9.8][6]). The holomorphic counterpart of (2.9) is Petersson's trace formula (cf. [Iw2, Theorem 9.6])

$$\delta_{mn} + 2\pi i^{-k}\sum_{D\mid c}\frac{1}{c}S_\psi(m,n,c)J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) = \frac{(k-2)!\sqrt{mn}}{(4\pi)^{k-1}}\sum_{f\in\mathcal{B}_k(D,\psi)}\overline{\rho_f(m)}\rho_f(n). \tag{2.10}$$

2.4. **Approximate functional equation.** For the proof of Proposition 1 we shall need to express L-functions as finite sums. Let g be a primitive (holomorphic or Maaß) cusp form of archimedean parameter μ, level D and arbitrary nebentypus. Define

$$\mu_1,\ \mu_2 := \begin{cases} i\mu, & -i\mu & \text{when } g \text{ is an even Maaß form of even weight;} \\ i\mu, & -i\mu+1 & \text{when } g \text{ is an even Maaß form of odd weight;} \\ i\mu+1, & -i\mu+1 & \text{when } g \text{ is an odd Maaß form of even weight;} \\ i\mu+1, & -i\mu & \text{when } g \text{ is an odd Maaß form of odd weight;} \\ i\mu, & i\mu+1 & \text{when } g \text{ is a holomorphic form.} \end{cases}$$

Observe that (2.4) implies

$$\Re\mu_1,\ \Re\mu_2 \geqslant -\theta = -\frac{7}{64}.$$

For $\Re s > 1$ the L-function of g is defined in terms of the Hecke eigenvalues $\lambda(n)$ as

$$L(g,s) := \sum_{n=1}^\infty \lambda(n)n^{-s}.$$

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[6] In [Iw2] the basis of the Eisenstein spectrum is indexed by the set $\{\mathfrak{a}\}$ of cusps of $\Gamma_0(D)$ which are singular with respect to ψ. The proof for the basis indexed by (2.1) is identical. Note also that in [Iw2] equation (9.15) should have the normalization factor $\frac{2}{\pi}$ instead of $\frac{4}{\pi}$, and in equation (B.49) a factor 4 is missing.

The completed L-function, given by

$$\Lambda(g,s) := D^{s/2} L_\infty(g,s) L(g,s), \qquad L_\infty(g,s) := \pi^{-s}\Gamma\left(\frac{s+\mu_1}{2}\right)\Gamma\left(\frac{s+\mu_2}{2}\right),$$

is entire and satisfies the functional equation

$$\Lambda(g,s) = \omega\overline{\Lambda}(g,1-\overline{s}), \tag{2.11}$$

for some $\omega=\omega(g)$ of modulus 1, cf. [DFI, (8.11)–(8.13), (8.17)–(8.19)]. This leads to the following representation of $L(g,s)$ as an essentially finite series for s on the critical line; the following approximate functional equation holds:

$$L\left(g,\frac{1}{2}+it\right) = \sum_{n=1}^{\infty}\frac{\lambda(n)}{n^{\frac{1}{2}+it}}V\left(\frac{n}{\sqrt{C(t)}}\right) + \kappa\sum_{n=1}^{\infty}\frac{\overline{\lambda(n)}}{n^{\frac{1}{2}-it}}\overline{V\left(\frac{n}{\sqrt{C(t)}}\right)} \tag{2.12}$$

for $t\in\mathbb{R}$ where $\kappa=\kappa(g,t)$ has absolute value 1,

$$C(t) := \frac{D}{(2\pi)^2}\left|\frac{1}{2}+it+\mu_1\right|\left|\frac{1}{2}+it+\mu_2\right| \tag{2.13}$$

is the analytic conductor, and V is a smooth function satisfying

$$x^j V^{(j)}(x) \ll_{j,k} (1+x)^{-k} \tag{2.14}$$

for each pair $(j,k)\in\mathbb{N}_0^2$, see [Ha, Theorem 2.1 and Remark 2.7].

3. Amplification

In the next three sections we give a proof of Theorem 2. The method is based on a paper by Bykovskiĭ [By]. Let f_0 be a primitive (holomorphic or Maaß) cusp form of Hecke eigenvalues $\lambda(n)$, archimedean parameter μ, level N and trivial nebentypus, and let χ be a primitive character modulo q for which we want to prove Theorem 2. We shall embed f_0 into the spectrum of $\Gamma_0(D)$ with trivial nebentypus, where D is an integer satisfying $[N,q]\mid D$ and $D>2q$; we take

$$D := 3[N,q]. \tag{3.1}$$

More precisely, we shall choose the bases $\mathcal{B}_k(D,1)$ and $\mathcal{B}(D,1)$ described in Section 2 in such a way that one of them contains the L^2-normalized version of $f_0(z)$:

$$f_1(z) := \frac{f_0(z)}{\langle f_0,f_0\rangle_D} = \frac{f_0(z)}{[\Gamma_0(q):\Gamma_0(D)]\,\langle f_0,f_0\rangle_q}.$$

Then (2.7)—applied for q in place of D—shows that

$$|\rho_{f_1}(1)|^2 \gg_\varepsilon \begin{cases}(4\pi)^{k-1}((k-1)!D)^{-1}(kD)^{-\varepsilon}, & \text{for} \quad f_1\in\mathcal{B}_k(D,1),\\ \cosh(\pi\mu)D^{-1}(D+|\mu|)^{-\varepsilon}, & \text{for} \quad f_1\in\mathcal{B}(D,1),\end{cases} \tag{3.2}$$

We shall consider an amplified square mean of the "fake" twisted L-functions[7]

$$\mathcal{L}(f\otimes\chi,s) := \sum_{n=1}^{\infty}\sqrt{n}\rho_f(n)\chi(n)n^{-s} \tag{3.3}$$

for f either in $\mathcal{B}_k(D,1)$ or $\mathcal{B}(D,1)$ and

$$\mathcal{L}(E_{\psi,\bar{\psi},f,t}\otimes\chi,s) := \sum_{n=1}^{\infty}\sqrt{n}\rho_{f,t}(n)\chi(n)n^{-s} \tag{3.4}$$

[7][By] considers true L-functions over the whole spectrum which is, technically speaking, incorrect as the spectrum includes old forms. Similarly, the "normalized orthonormal basis" considered at the bottom of [By, p.925] is problematic as the first Fourier coefficient vanishes for old forms. We avoid these troubles by a more careful setup here and in Sections 2.1–2.2.

for ψ any character modulo D, $f\in\mathcal{B}(\psi,\bar\psi)$ and $t\in\mathbb{R}$. The justification comes from (2.6): apart from invertible Euler factors at primes dividing D,

$$L(f_0\otimes\chi,s)\approx\sum_{n=1}^{\infty}\lambda(n)n^{-s},$$

hence for $\Re s=\frac12$ we have

$$|\mathcal{L}(f_1\otimes\chi,s)|\gg_\varepsilon D^{-\varepsilon}|\rho_{f_1}(1)||L(f_0\otimes\chi,s)|. \tag{3.5}$$

For integers $0\leqslant b<a$ let us define

$$\phi_{a,b}(x):=i^{b-a}J_a(x)x^{-b}. \tag{3.6}$$

In order to satisfy the decay conditions for Kuznetsov's trace formula, we assume $b\geqslant 2$. If $a\equiv b \pmod 2$, then using [GR, 6.574.2] it is straightforward to verify that

$$\begin{aligned}\dot\phi_{a,b}(k)&=\frac{b!}{2^{b+1}\pi}\prod_{j=0}^{b}\left\{\left(\frac{(1-k)i}{2}\right)^2+\left(\frac{a+b}{2}-j\right)^2\right\}^{-1}\asymp_{a,b}\ \pm k^{-2b-2},\\ \tilde\phi_{a,b}(t)&=\frac{b!}{2^{b+1}\pi}\prod_{j=0}^{b}\left\{t^2+\left(\frac{a+b}{2}-j\right)^2\right\}^{-1}\asymp_{a,b}\ (1+|t|)^{-2b-2}\end{aligned} \tag{3.7}$$

with $\dot\phi$ and $\tilde\phi$ as in (2.8). In particular,

$$\begin{aligned}&\dot\phi_{a,b}(k)>0\quad\text{for}\quad 2\leqslant k\leqslant a-b,\\ &\tilde\phi_{a,b}(t)>0\quad\text{for}\quad \text{all possible spectral parameters } t,\end{aligned} \tag{3.8}$$

since $|\Im t|<\frac12$. For

$$\tau\in\mathbb{R},\qquad u\in\mathbb{C},\qquad k\in\{2,4,6,\dots\},\qquad (\ell,D)=1$$

let us define the quantities

$$\begin{aligned}\mathcal{Q}_k^{\rm holo}(\ell)&:=\frac{i^k(k-2)!}{2\pi(4\pi)^{k-1}}\sum_{f\in\mathcal{B}_k(D,1)}\lambda_f(\ell)\mathcal{L}(f\otimes\chi,u+i\tau)\overline{\mathcal{L}(f\otimes\chi,\overline{u}+i\tau)}\\ \mathcal{Q}(\ell)&:=\sum_{k\geqslant 2\text{ even}}\dot\phi(k)2(k-1)i^{-k}\mathcal{Q}_k^{\rm holo}(\ell)\\ &\quad+\sum_{f\in\mathcal{B}(D,1)}\tilde\phi(t_f)\frac{4\pi}{\cosh(\pi t_f)}\lambda_f(\ell)\mathcal{L}(f\otimes\chi,u+i\tau)\overline{\mathcal{L}(f\otimes\chi,\overline{u}+i\tau)}\\ &\quad+\sum_{\substack{\psi\bmod D\\ f\in\mathcal{B}(\psi,\bar\psi)}}\int_{-\infty}^{\infty}\tilde\phi(t)\frac{1}{\cosh(\pi t)}\lambda_{\psi,\bar\psi,t}(\ell)\mathcal{L}(E_{\psi,\bar\psi,f,t}\otimes\chi,u+i\tau)\overline{\mathcal{L}(E_{\psi,\bar\psi,f,t}\otimes\chi,\overline{u}+i\tau)}\,dt\end{aligned}$$

with the notation (2.8) and with $\phi:=\phi_{20,2}$, cf. (3.6).

For $u=\frac12+\varepsilon$ and $k\geqslant 4$ we shall show in the next section

$$\begin{aligned}\mathcal{Q}_k^{\rm holo}(\ell)&\ll_\varepsilon\left(\frac{1}{\sqrt{\ell}}+\left(\frac{\ell^{\frac14}(N,q)}{q^{\frac12}N^{\frac12}}+\frac{\ell^{\frac12}(N,q)^{\frac32}}{q^{\frac12}N}\right)\left(\frac{1+|\tau|}{k}+1\right)\right)((1+|\tau|)D\ell)^{\varepsilon},\\ \mathcal{Q}(\ell)&\ll_\varepsilon\left(\frac{1}{\sqrt{\ell}}+\left(\frac{\ell^{\frac14}(N,q)}{q^{\frac12}N^{\frac12}}+\frac{\ell^{\frac12}(N,q)^{\frac32}}{q^{\frac12}N}\right)(1+|\tau|)\right)((1+|\tau|)D\ell)^{\varepsilon},\end{aligned} \tag{3.9}$$

with implied constants depending only on ε. Theorem 2 then follows by standard amplification: let us define the amplifier

$$x(\ell) := \begin{cases} \overline{\lambda(\ell)} & \text{for} \quad L \leqslant \ell \leqslant 2L, \;\; (\ell, D) = 1, \\ 0 & \text{else}, \end{cases} \tag{3.10}$$

where L is some parameter to be chosen in a moment. Let ω be a smooth cut-off function supported on $[1/2, 3]$. Then

$$\begin{aligned} \sum_{\substack{(\ell,D)=1 \\ \ell \sim L}} |\lambda(\ell)|^2 &\gg_\omega \frac{1}{2\pi i} \int_{(2)} L^{(D)}(f_0 \otimes \overline{f_0}, s) \hat{\omega}(s) L^s ds \\ &\gg_\varepsilon L(q(1+|\mu|)D)^{-\varepsilon} + O_\varepsilon\left(q^\varepsilon (L(1+|\mu|)N)^{\frac{1}{2}+\varepsilon}\right), \end{aligned}$$

where the superscript (D) indicates that the Euler factors of the Rankin–Selberg L-function at the primes dividing D have been omitted. The lower bound for the residue follows from [HL], while the error term uses the standard (convexity) bounds for the symmetric square L-function on the line $\Re s = \frac{1}{2} + \varepsilon$. Therefore,

$$\sum_{\ell} x(\ell)\lambda(\ell) = \sum_{\substack{(\ell,D)=1 \\ \ell \sim L}} |\lambda(\ell)|^2 \gg_\varepsilon L(LD)^{-\varepsilon}, \tag{3.11}$$

provided $L \geqslant q^\varepsilon((1+|\mu|)N)^{1+\varepsilon}$. Assume first that f_0 is a Maaß cusp form of weight zero or a holomorphic cusp form of weight 2. Then by (3.5), (3.2), (3.7) with $b = 2$, (3.8) and (3.11), we obtain

$$\begin{aligned} &\frac{L^2(LD)^{-\varepsilon}}{(1+|\mu|)^{6+\varepsilon} D} \left| L\left(f_0 \otimes \chi, \frac{1}{2} + \varepsilon + i\tau\right)\right|^2 \ll_\varepsilon \\ &\qquad \sum_{\substack{k \geqslant 2 \text{ even} \\ f \in \mathcal{B}_k(D,1)}}\sum |\dot{\phi}(k)| \frac{(k-1)!}{\pi(4\pi)^{k-1}} \left|\sum_\ell x(\ell)\lambda_f(\ell)\right|^2 \left|\mathcal{L}\left(f \otimes \chi, \frac{1}{2} + \varepsilon + i\tau\right)\right|^2 \\ &\qquad + \sum_{f \in \mathcal{B}(D,1)} \tilde{\phi}(t_f) \frac{4\pi}{\cosh(\pi t_f)} \left|\sum_\ell x(\ell)\lambda_f(\ell)\right|^2 \left|\mathcal{L}\left(f \otimes \chi, \frac{1}{2} + \varepsilon + i\tau\right)\right|^2 \\ &\qquad + \sum_{\substack{\psi \bmod D \\ f \in \mathcal{B}(\psi, \bar{\psi})}}\sum \int_{-\infty}^{\infty} \tilde{\phi}(t) \frac{1}{\pi\cosh(\pi t)} \left|\sum_\ell x(\ell)\lambda_{\psi,\bar{\psi},t}(\ell)\right|^2 \left|\mathcal{L}\left(E_{\psi,\bar{\psi},f,t} \otimes \chi, \frac{1}{2} + \varepsilon + i\tau\right)\right|^2 dt, \end{aligned}$$

so that by (2.2) and (3.8)

$$\begin{aligned} &\frac{L^2(LD)^{-\varepsilon}}{(1+|\mu|)^{6+\varepsilon} D} \left| L\left(f_0 \otimes \chi, \frac{1}{2} + \varepsilon + i\tau\right)\right|^2 \ll_\varepsilon \\ &\qquad \sum_{\ell_1,\ell_2} \left|x(\ell_1)x(\ell_2)\right| \sum_{d \mid (l_1,l_2)} \left\{ \left|\mathcal{Q}\left(\frac{\ell_1\ell_2}{d^2}\right)\right| + \sum_{k \geqslant 20 \text{ even}} 4k \left|\dot{\phi}_0(k)\right| \left|\mathcal{Q}_k^{\text{holo}}\left(\frac{\ell_1\ell_2}{d^2}\right)\right| \right\}. \end{aligned}$$

Now we substitute (3.9). Note that the k-sum converges by (3.7). Changing the order of summation, we get the bound

$$\begin{aligned}\ll_\varepsilon ((1+|\tau|)LD)^\varepsilon \Bigg\{ &\sum_d \sum_{\ell_1,\ell_2} (\ell_1\ell_2)^{-\frac12} |x(d\ell_1)x(d\ell_2)| \\ &+ \frac{(1+|\tau|)(N,q)}{q^{\frac12}N^{\frac12}} \sum_d \sum_{\ell_1,\ell_2} (\ell_1\ell_2)^{\frac14} |x(d\ell_1)x(d\ell_2)| \\ &+ \frac{(1+|\tau|)(N,q)^{\frac32}}{q^{\frac12}N} \sum_d \sum_{\ell_1,\ell_2} (\ell_1\ell_2)^{\frac12} |x(d\ell_1)x(d\ell_2)| \Bigg\}.\end{aligned}$$

In each term we have, by Cauchy–Schwarz ($a \in \mathbb{R}$),

$$\begin{aligned}\sum_d \sum_{\ell_1,\ell_2} (\ell_1\ell_2)^a |x(d\ell_1)x(d\ell_2)| &= \sum_d \left(\sum_\ell \ell^a |x(d\ell)|\right)^2 \leqslant \sum_d \left(\sum_{\ell \leqslant 2L} \ell^{2a}\right)\left(\sum_\ell |x(d\ell)|^2\right) \\ &= \left(\sum_{\ell\leqslant 2L} \ell^{2a}\right) \sum_\ell \tau(\ell)|x(\ell)|^2 \ll_a (1+L^{2a+1}) \sum_\ell \tau(\ell)|x(\ell)|^2,\end{aligned}$$

so that

$$\begin{aligned}&\frac{L^2(LD)^{-\varepsilon}}{(1+|\mu|)^{6+\varepsilon}D} \left| L\left(f_0\otimes\chi, \frac12+\varepsilon+i\tau\right)\right|^2 \ll_\varepsilon \\ &\qquad ((1+|\tau|)LD)^\varepsilon \left(1 + \frac{L^{\frac32}(N,q)}{q^{\frac12}N^{\frac12}}(1+|\tau|) + \frac{L^2(N,q)^{\frac32}}{q^{\frac12}N}(1+|\tau|)\right) \sum_\ell \tau(\ell)|x(\ell)|^2,\end{aligned}$$

This yields, by (3.1), (3.10) and (2.3),

$$\begin{aligned}&\left| L\left(f_0\otimes\chi, \frac12+\varepsilon+i\tau\right)\right|^2 \ll_\varepsilon \\ &\quad (1+|\mu|)^6 \left(\frac{qN}{L(N,q)} + L^{\frac12}q^{\frac12}N^{\frac12}(1+|\tau|) + Lq^{\frac12}(N,q)^{\frac12}(1+|\tau|)\right)((1+|\tau|+|\mu|)Nq)^\varepsilon,\end{aligned}$$

provided $L \geqslant q^\varepsilon((1+|\mu|)N)^{1+\varepsilon}$. For such L, the second term in the parenthesis is dominated by the third one which motivates our choice

$$L := \frac{q^{\frac14}N^{\frac12}}{(N,q)^{\frac34}(1+|\tau|)^{\frac12}} + q^\varepsilon(N(1+|\mu|))^{1+\varepsilon}.$$

We obtain

$$\begin{aligned}&L\left(f_0\otimes\chi, \frac12+\varepsilon+i\tau\right) \ll_\varepsilon \\ &(1+|\mu|)^3 \left((1+|\tau|)^{\frac14}N^{\frac14}q^{\frac38}(N,q)^{-\frac18} + (1+|\tau|)^{\frac12}(1+|\mu|)^{\frac12}N^{\frac12}(N,q)^{\frac14}q^{\frac14}\right)((1+|\tau|+|\mu|)Nq)^\varepsilon.\end{aligned}$$

By the functional equation and the Phragmén–Lindelöf convexity principle, we obtain Theorem 2 in the non-holomorphic case as well as in the case when f_0 is holomorphic of weight 2. Analogously, if f_0 is holomorphic of (even) weight $k \geqslant 4$, we get

$$\frac{L^2(kLD)^{-\varepsilon}}{kD}\left|L\left(f_0\otimes\chi,\frac{1}{2}+\varepsilon+i\tau\right)\right|^2 \ll_\varepsilon \sum_{\ell_1,\ell_2}\left|x(\ell_1)x(\ell_2)\right|\sum_{d|(l_1,l_2)}\left|\mathcal{Q}_k^{\mathrm{holo}}\left(\frac{\ell_1\ell_2}{d^2}\right)\right|$$

$$\ll_\varepsilon ((1+|\tau|)LD)^\varepsilon\left(1+\frac{L^{\frac{3}{2}}(N,q)}{q^{\frac{1}{2}}N^{\frac{1}{2}}}\left(\frac{1+|\tau|}{k}+1\right)+\frac{L^2(N,q)^{\frac{3}{2}}}{q^{\frac{1}{2}}N}\left(\frac{1+|\tau|}{k}+1\right)\right)\sum_\ell \tau(\ell)|x(\ell)|^2,$$

provided $L \geqslant q^\varepsilon (kN)^{1+\varepsilon}$. Choosing

$$L := \frac{q^{\frac{1}{4}}N^{\frac{1}{2}}k^{\frac{1}{2}}}{(N,q)^{\frac{3}{4}}(1+|\tau|+k)^{\frac{1}{2}}}+q^\varepsilon(kN)^{1+\varepsilon}$$

and using (3.1), (3.10) and (2.3), we obtain

$$L\left(f_0\otimes\chi,\frac{1}{2}+\varepsilon+i\tau\right)\ll_\varepsilon$$
$$k^{\frac{1}{2}}\left((|\tau|+k)^{\frac{1}{4}}k^{-\frac{1}{4}}N^{\frac{1}{4}}q^{\frac{3}{8}}(N,q)^{-\frac{1}{8}}+(|\tau|+k)^{\frac{1}{2}}N^{\frac{1}{2}}(N,q)^{\frac{1}{4}}q^{\frac{1}{4}}\right)((1+|\tau|)kNq)^\varepsilon.$$

This completes the proof of Theorem 2.

4. Variations on a theme of Bykovskiĭ

In order to show (3.9), we perform the following steps, cf. [By, Section 5].

Step 0. For later purposes let us define, for $u,s\in\mathbb{C}$, $\tau,x\in\mathbb{R}$, $\eta_{1,2}\in\{\pm1\}$ and $\phi_{a,b}$ as in (3.6),

$$E_{u,\tau}^{\eta_1,\eta_2}(s) := \begin{cases} -\exp(\eta_1\pi i(s/2+u)), & \text{for} \quad \eta_1=\eta_2,\\ \exp(\eta_1\pi\tau), & \text{for} \quad \eta_1\neq\eta_2,\end{cases}$$

and

$$\Xi_{u,\tau}^{\eta_1,\eta_2}(x) := \frac{1}{2\pi i}\int_{(\sigma)} E_{u,\tau}^{\eta_1,\eta_2}(s)\Gamma\left(1-\frac{s}{2}-u-i\tau\right)\Gamma\left(1-\frac{s}{2}-u+i\tau\right)\widehat{\phi}_{a,b}(s)2^{1-s}x^{-\frac{s}{2}}\,ds. \tag{4.1}$$

The Mellin transform of $\phi_{a,b}$ equals [GR, 6.561.14]

$$\widehat{\phi}_{a,b}(s) = i^{b-a}2^{s-b-1}\Gamma\left(\frac{a-b+s}{2}\right)\left(\Gamma\left(\frac{2+a+b-s}{2}\right)\right)^{-1}. \tag{4.2}$$

Thus the integrand in (4.1) is holomorphic and by Stirling's formula the integral converges absolutely if

$$b-a \;<\; \sigma \;<\; 2-2\Re u \;<\; 1+b. \tag{4.3}$$

Moreover, in this range we have, uniformly in a, τ, and $\Im u$,

$$\Xi_{u,\tau}^{\eta_1,\eta_2}(x)\ll_{b,\sigma,\Re u}$$
$$x^{-\frac{\sigma}{2}}\int_{-\infty}^{\infty}(a+|t|)^{\sigma-1-b}\left\{\left(1+\left|\frac{t}{2}+\Im(u+i\tau)\right|\right)\left(1+\left|\frac{t}{2}+\Im(u-i\tau)\right|\right)\right\}^{\frac{1}{2}-\frac{\sigma}{2}-\Re u}dt.$$

Breaking the integration into $|t|\leqslant 4(1+|\Im u|+|\tau|)$ and $|t|>4(1+|\Im u|+|\tau|)$ we find, for integers $0\leqslant b\leqslant 2<a$ and σ satisfying (4.3),

$$\Xi_{u,\tau}^{\eta_1,\eta_2}(x)\ll_{\sigma,\Re u}$$
$$x^{-\frac{\sigma}{2}}\begin{cases} a^{\sigma-1-b}(1+|\Im u|+|\tau|)^{2-\sigma-2\Re u}+(1+|\Im u|+|\tau|)^{1-2\Re u-b}, & \text{for} \quad \sigma<1+b,\\ a^{\sigma-b}(1+|\Im u|+|\tau|)^{1-\sigma-2\Re u}+(1+|\Im u|+|\tau|)^{1-2\Re u-b}, & \text{for} \quad \sigma<1-2\Re u.\end{cases}$$

In particular, for $u=1/2+\varepsilon$ we obtain

$$\Xi_{u,\tau}^{\eta_1,\eta_2}(x) \ll_\varepsilon x^{-\frac{1}{2}+2\varepsilon}(1+|\tau|)^{2\varepsilon}, \tag{4.4}$$

$$\Xi_{u,\tau}^{\eta_1,\eta_2}(x) \ll_\varepsilon x^{\frac{1}{2}+\varepsilon}\left(\frac{1+|\tau|}{a}+1\right), \tag{4.5}$$

upon choosing $\sigma=1-4\varepsilon$ and $\sigma=-1-2\varepsilon$, respectively, while for $1/2<\Re u<(a-b+1)/2-\varepsilon$ we have

$$\Xi_{u,\tau}^{\eta_1,\eta_2}(x) \ll_{a,\tau,\Re u,\varepsilon} x^{\frac{a-b}{2}-\varepsilon} \tag{4.6}$$

upon choosing $\sigma=b-a+2\varepsilon$. For $\alpha\in\mathbb{R}$ let

$$\zeta_{(\alpha)}(s) := \sum_{n+\alpha>0}(n+\alpha)^{-s}$$

be the Hurwitz zeta-function. It satisfies a functional equation

$$\zeta_{(\alpha)}(s) = (2\pi)^{s-1}\Gamma(1-s)\left\{-ie\left(\frac{s}{4}\right)\zeta^{(\alpha)}(1-s)+ie\left(-\frac{s}{4}\right)\zeta^{(-\alpha)}(1-s)\right\}, \tag{4.7}$$

where

$$\zeta^{(\alpha)}(s) := \sum_{n=1}^\infty e(\alpha n)n^{-s}.$$

Step 1. Let us first assume $5/4<\Re u<3/2$. By combining (2.5) with Petersson's (resp. Kuznetsov's) trace formula (2.10) (resp. (2.9)) we obtain the following expressions for $\mathcal{Q}_k^{\mathrm{holo}}(\ell)$ (resp. $\mathcal{Q}(\ell)$), cf. [By, (5.3)]:

$$\begin{aligned} &\frac{\alpha_{1/2+i\tau}^{(\chi)}(\ell)}{2\pi i^{-k}\ell^u}\prod_{p\mid q}\left(1-\frac{1}{p^{2u}}\right)\zeta(2u)\\ &+\sum_{D\mid c}\frac{1}{c}\sum_{m_1,m_2}\frac{S(m_1,m_2,-\ell;c)}{(m_1m_2)^u}\left(\frac{m_2}{m_1}\right)^{i\tau}\chi(m_1)\overline{\chi(m_2)}\phi\left(\frac{4\pi\sqrt{m_1m_2\ell}}{c}\right),\end{aligned} \tag{4.8}$$

where

$$\alpha_{1/2+i\tau}^{(\chi)}(\ell) := \sum_{\ell_1\ell_2=\ell}\chi(\ell_1)\overline{\chi(\ell_2)}\left(\frac{\ell_2}{\ell_1}\right)^{i\tau},$$

$$S(m_1,m_2,m_3;c) := \frac{1}{c}\sum_{a_1,a_2,a_3\,(c)} e\left(\frac{a_1a_2a_3+m_1a_1+m_2a_2+m_3a_3}{c}\right),$$

and

$$\phi := \begin{cases} J_{k-1}=\phi_{k-1,0} & \text{if } f \text{ is holomorphic of weight } k\geqslant 4;\\ \phi_{20,2} & \text{otherwise.}\end{cases} \tag{4.9}$$

The diagonal term in the first line of (4.8) only appears in the holomorphic case. The sum in the second line converges absolutely once $\Re u>5/4$. In the following we transform the off-diagonal term further.

Step 2. We open ϕ and write it as an inverse Mellin transform

$$\phi\left(\frac{4\pi\sqrt{m_1m_2\ell}}{c}\right) = \frac{1}{2\pi i}\int_{(\sigma)}\widehat{\phi}(s)\left(\frac{c}{4\pi\sqrt{m_1m_2\ell}}\right)^s ds.$$

By (4.2) the integrand is holomorphic and the integral converges absolutely if $-3<\sigma=\Re s<0$ in both the holomorphic (note $k\geqslant 4$) and the non-holomorphic case; the m_1,m_2-sum converges absolutely if $\Re u+\sigma/2>1$, and the c-sum converges absolutely if $\sigma<-1/2$ (Weil's bound, cf. [By, Lemmata 1 and 3]). If we impose $2-2\Re u<\sigma<-1/2$, we can interchange the s-integration and the m_1,m_2-sum. Now splitting into residue classes modulo c, we write the m_1,m_2-sum as a linear combination of a product of two Hurwitz ζ-functions getting

$$\sum_{D|c}\frac{1}{c^{2u+1}}\frac{1}{2\pi i}\int_{(\sigma)}\widehat{\phi}(s)(4\pi\sqrt{\ell})^{-s}\sum_{b_1,b_2\,(c)}S(b_1,b_2,-\ell;c)\chi(b_1)\overline{\chi(b_2)}$$
$$\times\,\zeta_{(\frac{b_1}{c})}\left(\frac{s}{2}+u+i\tau\right)\zeta_{(\frac{b_2}{c})}\left(\frac{s}{2}+u-i\tau\right)ds.$$

By standard bounds for the Hurwitz ζ-function the s-integral and the c-sum converge absolutely if $\Re u>5/4$ and $-3<\sigma<0$.

Step 3. We shift the integration to any line $-3<\sigma<-2\Re u$. By [By, Lemma 6] if $\tau\neq 0$ and by [By, Lemma 2] if $\tau=0$, we pick up poles only if $\frac{c}{q}\mid\ell$. Since $(\ell,D)=1$, $D\mid c$ and $\frac{D}{q}>1$, this does not happen[8]. Now we apply the functional equation (4.7) for the two Hurwitz ζ-functions[9], and write them as Dirichlet series getting (cf. [By, (5.8)])

$$\sum_{D|c}\frac{(2\pi)^{2u-2}}{2c^{2u+1}}\sum_{m_1,m_2\in\mathbb{Z}\setminus\{0\}}|m_1|^{u-1+i\tau}|m_2|^{u-1-i\tau}\sum_{b_1,b_2(c)}S(b_1,b_2,-\ell;c)\chi(b_1)\overline{\chi(b_2)}e\left(\frac{m_1b_1+m_2b_2}{c}\right)$$
$$\times\,\Xi_{u,\tau}^{\mathrm{sgn}(m_1),\mathrm{sgn}(m_2)}\left(\frac{\ell}{|m_1m_2|}\right),$$

where $\Xi_{u,\tau}^{\mathrm{sgn}(m_1),\mathrm{sgn}(m_2)}$ with ϕ as in (4.9) was defined in (4.1). This expression converges absolutely if $\Re u>5/4$. Note that when we apply (4.4)–(4.6) in the following, we have $(a,b)=(k-1,0)$ with $k\geqslant 4$ or $(a,b)=(20,2)$.

Step 4. We transform the b_1,b_2-sum by [By, Lemma 2] obtaining

$$\sum_{D|c}\frac{(2\pi)^{2u-2}}{2c^{2u-1}q}\sum_{\substack{m_1m_2\neq 0\\ m_1m_2\equiv\ell\,(c/q)}}|m_1|^{u-1+i\tau}|m_2|^{u-1-i\tau}\sideset{}{^*}\sum_{d\,(q)}\bar{\chi}\left(m_1+\frac{c}{q}\bar{d}\right)\chi\left(m_2+\frac{m_1m_2-\ell}{c/q}d\right)$$
$$\times\,\Xi_{u,\tau}^{\mathrm{sgn}(m_1),\mathrm{sgn}(m_2)}\left(\frac{\ell}{|m_1m_2|}\right).$$

We will see in a moment that this term can be analytically continued to $\Re u>1/2$. Let us start with the terms $m_1m_2\neq\ell$. Their contribution equals

$$\frac{1}{4\pi q}\left(\frac{2\pi}{q}\right)^{2u-1}\sum_{\substack{m_1m_2-n_1n_2=\ell\\ m_1m_2n_1n_2\neq 0\\ D/q|n_1}}\frac{X}{|m_1m_2|^{1/2}}\left|\frac{m_1m_2}{n_1^2}\right|^{u-1/2}\left|\frac{m_1}{m_2}\right|^{i\tau}\Xi_{u,\tau}^{\mathrm{sgn}(m_1),\mathrm{sgn}(m_2)}\left(\frac{\ell}{|m_1m_2|}\right),\tag{4.10}$$

where

$$X:=\sideset{}{^*}\sum_{d\,(q)}\bar{\chi}(m_1+n_1\bar{d})\chi(m_2+n_2d)\ll_\varepsilon q^{1/2+\varepsilon}(m_1,m_2,q)^{1/2}(n_1n_2,q)^{1/2}.\tag{4.11}$$

This estimate strengthens [By, Lemma 4] and follows essentially from the Riemann Hypothesis over finite fields. We provide a detailed proof in the next section, see Proposition 2. The condition $(\ell,q)=1$ is crucial here and in the sequel. By (4.6), the term (4.10) is holomorphic in $1/2<\Re u<3/2$. Let us take $u:=1/2+\varepsilon$. We split the sum in (4.10) into two parts: $|m_1m_2|>\ell$, $|m_1m_2|<\ell$. Notice that $m_1m_2=-\ell$ cannot happen, since $m_1m_2\equiv\ell\pmod{D/q}$ and (3.1) would then imply $(2\ell,D)\geqslant D/q>2$ which contradicts $(\ell,D)=1$.

[8] It can be shown [By, (5.10)] that the residues in the case $\frac{c}{q}\mid\ell$ would be harmless.

[9] i.e., we apply Poisson summation to both m_1 and m_2 in (4.8)

Using (4.5), the terms $|m_1m_2| > \ell$ contribute at most

$$\ll_\varepsilon (\ell q)^\varepsilon \left(\frac{\ell}{q}\right)^{\frac{1}{2}} \left(\frac{1+|\tau|}{a}+1\right) \sum_{\substack{d_1,d_2\mid q\\(d_1,d_2)=1}} (d_1d_2)^{\frac{1}{2}} \sum_{\substack{m>\ell\\ m\equiv 0\,(d_1^2)\\ m\equiv \pm\ell\,([d_2,D/q])}} \frac{1}{m^{1+\varepsilon}},$$

where $a := 20$ in the non-holomorphic case and $a := k-1$ in the holomorphic case. The smallest element in the arithmetic progression given by the inner sum is at least $\max(\ell, d_1^2, \frac{1}{2}[d_2, D/q])$, therefore the above is at most

$$\begin{aligned}&\ll_\varepsilon (\ell q)^\varepsilon \left(\frac{\ell}{q}\right)^{\frac{1}{2}} \left(\frac{1+|\tau|}{a}+1\right) \left(\sum_{d\mid q} \frac{d^{\frac{1}{2}}}{[d,D/q]} + \sum_{d_1,d_2\mid q} \frac{(d_1d_2)^{\frac{1}{2}}}{\ell^{\frac{1}{4}} d_1^{\frac{1}{2}} [d_2,D/q]^{\frac{1}{2}}}\right)\\ &\ll_\varepsilon (\ell q)^\varepsilon \left(\frac{\ell}{q}\right)^{\frac{1}{2}} \left(\frac{1+|\tau|}{a}+1\right) \left(\frac{(N,q)^{\frac{3}{2}}}{N} + \frac{(N,q)}{\ell^{\frac{1}{4}} N^{\frac{1}{2}}}\right).\end{aligned} \tag{4.12}$$

In the last step we used the definition of D (cf. (3.1)).

By (4.4), the terms $|m_1m_2| < \ell$ contribute at most

$$\begin{aligned}&\ll_\varepsilon \frac{(\ell q(1+|\tau|))^\varepsilon}{(\ell q)^{\frac{1}{2}}} \sum_{\substack{d_1,d_2\mid q\\(d_1,d_2)=1}} (d_1d_2)^{\frac{1}{2}} \sum_{\substack{0<m<\ell\\ m\equiv 0\,(d_1^2)\\ m\equiv \pm\ell\,([d_2,D/q])}} 1\\ &\ll_\varepsilon \frac{(\ell q(1+|\tau|))^\varepsilon}{(\ell q)^{\frac{1}{2}}} \sum_{\substack{d_1,d_2\mid q\\(d_1,d_2)=1}} (d_1d_2)^{\frac{1}{2}} \left(\frac{\ell}{[d_1^2d_2, D/q]}+1\right)\\ &\ll_\varepsilon (\ell q(1+|\tau|))^\varepsilon \left(\left(\frac{\ell}{q}\right)^{\frac{1}{2}} \frac{(N,q)^{\frac{3}{2}}}{N} + \frac{1}{\sqrt{\ell}}\right).\end{aligned} \tag{4.13}$$

Finally the contribution of the terms $m_1m_2 = \ell$ is

$$\begin{aligned}&\sum_{D\mid c} \frac{(2\pi)^{2u-2}}{2c^{2u-1}q\ell^{1-u}} \sum_{m_1m_2=\ell} \chi(m_2) \left|\frac{m_1}{m_2}\right|^{i\tau} \sum_{a\,(q)}^{*} \bar\chi\left(m_1 + \frac{c}{q}\bar a\right) \left(\Xi_{u,\tau}^{1,1}(1) + \Xi_{u,\tau}^{-1,-1}(1)\right)\\ &= \frac{(2\pi)^{2u-2}\left(\Xi_{u,\tau}^{1,1}(1) + \Xi_{u,\tau}^{-1,-1}(1)\right)}{2D^{2u-1}q\ell^{1-u}} \sum_{m_1m_2=\ell} \chi(m_2)\left|\frac{m_1}{m_2}\right|^{i\tau} \sum_c \frac{1}{c^{2u-1}} \sum_{a\,(q)}^{*} \bar\chi\left(m_1+\frac{D}{q}\bar a c\right).\end{aligned} \tag{4.14}$$

We write $r := (D/q, q)$. Then the c, a-sum equals

$$\left(\frac{q}{r}\right)^{1-2u} \sum_{b\,(q/r)} \sum_{a\,(q)}^{*} \bar\chi(m_1 + r\bar a b)\zeta_{(\frac{b}{q/r})}(2u-1)$$

which is holomorphic for $\mathbb{C}\setminus\{1/2\}$. By the functional equation (4.7), this is for $\Re u > 1/2$

$$\begin{aligned}&-i\left(\frac{q}{r}\right)^{1-2u} (2\pi)^{2u-2}\Gamma(2-2u) e\left(\frac{2u-1}{4}\right) \sum_n \frac{1}{n^{2-2u}} \sum_{b\,(q/r)} \sum_{a\,(q)}^{*} \bar\chi(m_1 + r\bar a b) e(brn/q)\\ &+i\left(\frac{q}{r}\right)^{1-2u} (2\pi)^{2u-2}\Gamma(2-2u) e\left(\frac{1-2u}{4}\right) \sum_n \frac{1}{n^{2-2u}} \sum_{b\,(q/r)} \sum_{a\,(q)}^{*} \bar\chi(m_1 + r\bar a b) e(-brn/q).\end{aligned}$$

The a, b-sum decomposes into Ramanujan sums,

$$\sum_b \sum_a \ldots = \sum_{\substack{d\,(q)\\ r\mid d}} \bar\chi(m_1+d) \sum_{a\,(q)}^{*} e\left(\pm\frac{adn}{q}\right) = \sum_{\substack{d\,(q)\\ r\mid d}} \bar\chi(m_1+d) \sum_{s\mid(dn,q)} s\mu\left(\frac{q}{s}\right),$$

showing that both n-sums equal

$$\sum_{\substack{d\,(q)\\ r\mid d}} \bar\chi(m_1+d)\sum_{s\mid q} s\mu\left(\frac{q}{s}\right)\sum_{\frac{s}{(d,s)}\mid n}\frac{1}{n^{2-2u}} = \zeta(2-2u)\sum_{\substack{d\,(q)\\ r\mid d}} \bar\chi(m_1+d)\sum_{s\mid q}\mu\left(\frac{q}{s}\right)\frac{(d,s)^{2-2u}}{s^{1-2u}}.$$

We substitute this back into (4.14), and obtain by (4.4) that this term for $u=1/2+\varepsilon$ is bounded by

$$\frac{(\ell q(1+|\tau|))^{\varepsilon}}{q\sqrt{\ell}}\sum_{d\,(q)}(d,q) \ll_{\varepsilon} \frac{(\ell q(1+|\tau|))^{\varepsilon}}{\sqrt{\ell}}. \tag{4.15}$$

Collecting the first line of (4.8), (4.12), (4.13), and (4.15), we arrive at (3.9) for $u=1/2+\varepsilon$.

5. A character sum estimate

In this section we state in more precise form the bound (4.11) and provide a detailed proof.

Proposition 2. *Let χ be a primitive character modulo q and let m_1, m_2, n_1, n_2 be arbitrary integers satisfying $(m_1m_2-n_1n_2,q)=1$. Then we have the uniform bound*[10]

$$X(m_1,m_2,n_1,n_2) := \sideset{}{^*}\sum_{a\,(q)} \bar\chi(m_1+n_1\bar a)\chi(m_2+n_2a) \ll q^{1/2}\tau(q)(m_1n_1^2,m_2n_2^2,q)^{1/2},$$

where the implied constant is absolute.

By the multiplicative nature of these sums it suffices to show that

$$|X(m_1,m_2,n_1,n_2)| \leqslant q^{1/2}(m_1n_1^2,m_2n_2^2,q)^{1/2}\times\begin{cases}2, & q=p^\beta \text{ for a prime } p>2;\\ 2^{5/2} & q=p^\beta \text{ for } p=2.\end{cases} \tag{5.1}$$

Case 1. First we discuss the case when $\beta=1$ (that is, when q is prime). We apply [IK, Theorem 11.23] with the parameters $n=1$, $\mathbb{F}:=\mathbb{F}_q$, and

$$f(x) := x(m_1x+n_1)^{d-1}(m_2+n_2x),$$

where $d>1$ is the order of χ. The only thing we have to check is that f is not a d-th power. If $d>2$ then f can only be a d-th power if $n_1=n_2=0$ in $\mathbb{F}$ in which case the displayed bound is trivial. If $d=2$ then f can only be a d-th power if $n_1=n_2=0$ or $m_1=m_2=0$ in $\mathbb{F}$ in which case the displayed bound (5.1) is again trivial. Otherwise (5.1) follows from [IK, Theorem 11.23].

Case 2. Now we discuss the case when $\beta>1$ is even, say $\beta=2\alpha$. We apply [IK, Lemma 12.2] for the rational functions

$$f(x) := x\frac{m_2+n_2x}{m_1x+n_1}, \qquad g(x) := 0.$$

Then

$$f'(x) = \frac{m_1n_2x^2+2n_1n_2x+m_2n_1}{(m_1x+n_1)^2},$$

therefore it suffices to show that the congruence

$$m_1n_2y^2+2n_1n_2y+m_2n_1 \equiv 0 \pmod{p^\alpha} \tag{5.2}$$

under the restriction

$$y(m_2+n_2y)(m_1y+n_1) \not\equiv 0 \pmod{p} \tag{5.3}$$

has at most $2(n_1,n_2,p^\alpha)$ solutions when $p>2$ and at most $4(n_1,n_2,p^\alpha)$ solutions when $p=2$. We can clearly assume that $(n_1,n_2,p^\alpha)<p^\alpha$ for otherwise the assertion is trivial. Let us first assume that $p>2$. If $p\mid m_1$ and $p\mid m_2$ then the condition $(m_1m_2-n_1n_2,q)=1$ shows that (5.2) has

[10] Note that $(m_1m_2-n_1n_2,q)=1$ implies $(m_1n_1^2,m_2n_2^2,q)=(m_1,m_2,q)(n_1^2,n_2^2,q)\mid(m_1,m_2,q)(n_1n_2,q)$.

no solution satisfying $p \nmid y$. Therefore, without loss of generality, we can assume that $p \nmid m_1$. We multiply both sides of (5.2) by m_1 to see that the congruence is equivalent to

$$n_2(m_1y+n_1)^2 \equiv n_1(n_1n_2-m_1m_2) \pmod{p^\alpha}.$$

By assumption, the parentheses on both sides are coprime with p, hence a solution can only exist if $p^\gamma \parallel n_1$ and $p^\gamma \parallel n_2$ for some $0 \leqslant \gamma \leqslant \alpha-1$, and then the number of solutions of (5.2) under (5.3) is at most $2p^\gamma = 2(n_1,n_2,p^\alpha)$ by the structure of the group $(\mathbb{Z}/p^{\alpha-\gamma})^\times$. For $p=2$ we adjust the above argument slightly. First of all, we can assume that $\alpha > 2$ for otherwise (5.2) trivially has at most 4 solutions. If $4 \mid m_1$ and $4 \mid m_2$ then the condition $(m_1m_2-n_1n_2,q)=1$ shows that (5.2) has no solution satisfying $2 \nmid y$. Therefore, without loss of generality, we can assume that $4 \nmid m_1$. We multiply both sides of (5.2) by m_1 to see that the congruence is equivalent to

$$n_2(m_1y+n_1)^2 \equiv n_1(n_1n_2-m_1m_2) \pmod{2^\alpha(m_1,2)}.$$

If $2 \mid n_1n_2$ then $2 \nmid m_1m_2$ and we conclude, similarly as in the case of $p>2$, that the number of solutions of (5.2) under (5.3) is at most $4(n_1,n_2,2^\alpha)$. If $2 \nmid n_1n_2$ then the number of solutions of the congruence

$$n_2x^2 \equiv n_1(n_1n_2-m_1m_2) \pmod{2^\alpha(m_1,2)}$$

is at most 4 while the map $\mathbb{Z}/2^\alpha \to \mathbb{Z}/2^\alpha(m_1,2)$ given by $y \mapsto m_1y+n_1$ is injective, hence the number of solutions of (5.2) under (5.3) is also at most 4.

Case 3. Finally we discuss the case when $\beta > 1$ is odd, say $\beta = 2\alpha+1$. We apply [IK, Lemma 12.3] for the rational functions

$$f(x) := x\frac{m_2+n_2x}{m_1x+n_1}, \qquad g(x) := 0.$$

Then

$$f'(x) = \frac{m_1n_2x^2+2n_1n_2x+m_2n_1}{(m_1x+n_1)^2}, \qquad f''(x) = \frac{2n_1(n_1n_2-m_1m_2)}{(m_1x+n_1)^3},$$

hence for $p \nmid 2n_1$ the bound (5.1) follows from the already proven fact that (5.2) under (5.3) has at most 2 solutions and for $p=2$ the bound (5.1) follows from the already proven fact that (5.2) under (5.3) has at most $4(n_1,n_2,p^\alpha)$ solutions. For $p \mid n_1$ $(p>2)$ it suffices to show that in any complete residue systems modulo p^α there are at most $2p^{-1}(n_1,n_2,p^{\alpha+1})$ solutions of the congruence

$$m_1n_2y^2+2n_1n_2y+m_2n_1 \equiv 0 \pmod{p^{\alpha+1}} \tag{5.4}$$

under (5.3). We can clearly assume that $(n_1,n_2,p^{\alpha+1}) < p^{\alpha+1}$ for otherwise the assertion is trivial. By the condition $(m_1m_2-n_1n_2,q)=1$ we have $p \nmid m_1$, hence (5.4) is equivalent to

$$n_2(m_1y+n_1)^2 \equiv n_1(n_1n_2-m_1m_2) \pmod{p^{\alpha+1}}.$$

By assumption, the parentheses on both sides are coprime with p, hence a solution of (5.4) can only exist if $p^\gamma \parallel n_1$ and $p^\gamma \parallel n_2$ for some $1 \leqslant \gamma \leqslant \alpha$, and then the number of solutions of (5.4) under (5.3) is at most $2p^\gamma$ by the structure of the group $(\mathbb{Z}/p^{\alpha+1-\gamma})^\times$. In particular, n_1 and n_2 are both divisible by p and the solutions of (5.4) under (5.3) form $2p^{\gamma-1} = 2p^{-1}(n_1,n_2,p^{\alpha+1})$ complete residue classes modulo p^α. This completes the proof of Proposition 2.

6. A shifted convolution problem

The main ingredient for the proof of Proposition 1 is following result:

Proposition 3. *Let $\lambda(n)$ be the Hecke eigenvalues of a primitive (holomorphic or weight zero Maaß) cusp form of level D, arbitrary nebentypus and archimedean parameter μ. Let N, P, H be real numbers greater than $1/2$ satisfying $HP^2 \leqslant N^{1-\varepsilon}$. For $1 \leqslant h \leqslant H$ let W_h be a smooth function* *supported on* $[\frac{1}{3}N,4N]^2$ *such that uniformly* $\|W_h^{(i,j)}\|_\infty \ll_{i,j} (P/N)^{i+j}$ *for all* $i,j\in\mathbb{N}_0$. *Then we have*

$$\sum_{h\leqslant H}\sum_{m_1-m_2=h}\lambda(m_1)\overline{\lambda(m_2)}W_h(m_1,m_2)\ll_{\mu,\varepsilon}(HN)^{\frac12}P^{\frac32}D^{\frac52}\left(P^{\frac12}+\left(\frac{N}{PH}\right)^{\theta}\left(\frac{H}{D}\right)^{\frac12}\right)(HNPD)^{\varepsilon}.$$

for all $\varepsilon>0$ *and* θ *as in* (2.4).

This is Theorem 2 in [Bl1] with $l_1=l_2=h_1=a(h)=1$, where we have made the dependence on the level D explicit. To this end, we just note that in equation (3.1) of [Bl1] we have $\Lambda\asymp Q^2/D$, and in the display following (3.20) of [Bl1] we have

$$\sum_{r\leqslant z}|A_y(r)|^2\ll_{\mu,\varepsilon}(D^4y(z+y))^{1+\varepsilon}$$

by (2.6) of [Bl1]. By the large sieve inequality [Bl1, Lemma 2.5], (3.21) of [Bl1] is bounded by

$$\ll_{\mu,\varepsilon}\frac{N^2\sqrt{H}Q^{\varepsilon}}{\Lambda QP}\int_{1/2}^{2H}\int_{1/2}^{\mathcal{M}}\int_{\mathcal{I}(Z,y)}\frac{\Xi}{Z}\left(Z+\frac{\sqrt{H}}{\sqrt{D}}\right)\left(Z+\frac{\sqrt{z}}{\sqrt{D}}\right)D^2\sqrt{y}(z+y)^{1/2}\,dz\,dy\,d\xi,$$

analogously to the first display on [Bl1, p.127]. After the same calculation as on [Bl1, p.127] we arrive at Proposition 3.

We shall need the following corollary:

Corollary 3. *Use the same notation as in Proposition 3, but let* $\tilde W_h$ *be a smooth function supported on* $[\frac12N,3N]$ *such that uniformly* $\|\tilde W_h^{(j)}\|_\infty\ll_j (P/N)^j$ *for all* $j\in\mathbb{N}_0$. *Then we have*

$$\sum_{h\leqslant H}\sum_{m}\lambda(m)\overline{\lambda(m-h)}\tilde W_h(m)\ll_{\mu,\varepsilon}(HN)^{\frac12}P^{\frac32}D^{\frac52}\left(P^{\frac12}+\left(\frac{N}{PH}\right)^{\theta}\left(\frac{H}{D}\right)^{\frac12}\right)(HNPD)^{\varepsilon}.$$

for all $\varepsilon>0$ *and* θ *as in* (2.4).

This follows immediately on choosing any smooth function ϕ supported on $[-N/P,N/P]$ satisfying $\phi(0)=1$ and $\|\phi^{(j)}\|_\infty\ll_j(P/N)^j$ for all $j\in\mathbb{N}_0$, and applying Proposition 3 with

$$W_h(x,y):=\tilde W_h(x)\phi(x-h-y).$$

7. Proof of Proposition 1

Let g be a primitive cusp form as in Proposition 1 and Theorem 3; let T, K be large parameters such that

$$T\geqslant 2(1+|\mu|)\qquad\text{and}\qquad T^{\frac23+\varepsilon}\leqslant K\leqslant T. \tag{7.1}$$

Let ψ be a smooth function, supported on $(T-\frac12K,T+2K)$ such that $\psi(x)=1$ if $x\in(T,T+K)$ and $\|\psi^{(j)}\|\ll_j K^{-j}$ for any $j\in\mathbb{N}_0$. By the approximate functional equation (2.12) we have

$$\int_0^\infty\psi(t)\left|L\left(g,\frac12+it\right)\right|^2dt\leqslant 4\int_0^\infty\psi(t)\left|\sum\frac{\lambda(n)}{n^{\frac12+it}}V\left(\frac{n}{\sqrt{C(t)}}\right)\right|^2dt \tag{7.2}$$

with $C(t)$ as in (2.13) satisfying

$$C(t)\asymp DT^2,\qquad \frac{\partial^j}{\partial t^j}C(t)\ll_j DT^{2-j} \tag{7.3}$$

and V as in (2.14). Hence up to a negligible error we can assume

$$n\leqslant\mathcal{N}:=(\sqrt{D}T)^{1+\varepsilon}. \tag{7.4}$$

With $K = T$ we get from Montgomery–Vaughan's variant of Hilbert's inequality (see e.g. [Br, Satz 4.4.3])

$$\int_T^{2T} \left| L\left(g, \frac{1}{2} + it\right) \right|^2 dt \ll T \sum_{n \leqslant \mathcal{N}} \frac{|\lambda(n)|^2}{n} + \sum_{n \leqslant \mathcal{N}} |\lambda(n)|^2 \ll_{\mu,\varepsilon} (\sqrt{D}T)^{1+\varepsilon}$$

by (2.3) and partial summation. This gives the second estimate of part b) of Proposition 1. Let us now open the square in (7.2). The diagonal term contributes

$$\ll \sum_{n \leqslant \mathcal{N}} \frac{|\lambda(n)|^2}{n} K \ll_{\mu,\varepsilon} K(TD)^{\varepsilon} \tag{7.5}$$

by (2.3). We write the off-diagonal term as

$$\sum_{N \leqslant \mathcal{N}} \sum_{n \asymp N} \sum_{\substack{h \neq 0 \\ h \leqslant n-1}} \lambda(n) \overline{\lambda(n-h)} \tilde{W}_{h,N}(n), \tag{7.6}$$

where N runs over dyadic integers and

$$\tilde{W}_{h,N}(x) := \frac{\rho_N(x)}{\sqrt{x(x-h)}} \int_0^\infty \psi(t) e\left(\frac{t}{2\pi} \log\left(1 - \frac{h}{x}\right)\right) V\left(\frac{x}{\sqrt{C(t)}}\right) \overline{V\left(\frac{x-h}{\sqrt{C(t)}}\right)} dt$$

for a smooth function ρ_N supported on $[\frac{1}{2}N, 3N]$ such that $\|\rho^{(j)}\| \ll_j N^{-j}$ for all $j \in \mathbb{N}_0$. By (2.14) and (7.3) we have

$$\frac{\partial^j}{\partial t^j} V\left(\frac{x}{\sqrt{C(t)}}\right) \ll_j T^{-j}$$

for all $j \in \mathbb{N}_0$; hence partial integration shows

$$\tilde{W}_{h,N}(x) \ll_j \frac{K}{\sqrt{x(x-h)}} \left(\left| \log\left(1 - \frac{h}{x}\right) \right| K \right)^{-j}$$

for all $j \in \mathbb{N}_0$ and $x \asymp N$. In particular, choosing j sufficiently large, we see that $\tilde{W}_{h,N}(x)$ is negligible unless

$$|h| \leqslant \mathcal{H} := \left(\frac{N}{K}\right)^{1+\varepsilon}, \tag{7.7}$$

and for such h we have $\sqrt{x(x-h)} \asymp N$. Furthermore, in this range we have

$$\frac{\partial^j}{\partial x^j} \tilde{W}_{h,N}(x) \ll_j \frac{K}{x} \left(\frac{1}{x} + \frac{hT}{x^2}\right)^j \ll_{j,\varepsilon} T^{\varepsilon} \frac{K}{N} \left(\frac{T}{Kx}\right)^j$$

for all $j \in \mathbb{N}_0$. Now we apply Proposition 3 with $H = \mathcal{H}$ and $P = T/K$ to the two inner sums in (7.6). Note that the condition $HP^2 \leqslant N^{1-\varepsilon}$ is satisfied if $K \geqslant T^{2/3+\varepsilon}$ which is ensured by (7.1). By Corollary 3, (7.7), and (7.4) we see that (7.6) is at most

$$\begin{aligned} &\ll_{\mu,\varepsilon} (DT)^{\varepsilon} \max_{N \leqslant \mathcal{N}} \left(\frac{K}{N} \left(\frac{N^2}{K}\right)^{1/2} \left(1 + \frac{T}{K}\right)^{3/2} D^{5/2} \left(\left(\frac{T}{K}\right)^{1/2} + \left(\frac{K^2}{T}\right)^{\theta} \left(\frac{N}{DK}\right)^{1/2} \right) \right) \\ &\ll_{\mu,\varepsilon} (DT)^{\varepsilon} \frac{T^2 D^{5/2}}{K^{3/2}} \left(1 + \frac{K^{2\theta}}{T^{\theta} D^{1/4}}\right). \end{aligned}$$

Together with (7.5) we see that for any K satisfying (7.1), we have

$$\int_T^{T+K} \left| L\left(g, \frac{1}{2} + it\right) \right|^2 dt \ll_{\mu,\varepsilon} \left(K + \frac{T^2 D^{5/2}}{K^{3/2}} \left(1 + \frac{K^{2\theta}}{T^{\theta} D^{1/4}}\right) \right) (TD)^{\varepsilon}.$$

This gives the remaining estimates of Proposition 1.

8. Proof of Theorems 3 and 1

From the functional equation (2.11) one can deduce the following essentially well-known lemma (see for example [Go2, p.294]):

Lemma 1. *For g as in Theorem 3 and for $\Re s = \frac{1}{2}$ we have*

$$L(g,s) \ll_\varepsilon ((1+|\mu|)D|s|)^\varepsilon \left(1 + \int_0^\infty |L(g, s+it)|^2 e^{-\frac{1}{2}t^2}\, dt\right)^{1/2}.$$

The second part of Theorem 3 now follows immediately from Lemma 1 and Proposition 1. The first part can be verified by checking the cases $D \leqslant |s|^{1/5}$, $|s|^{1/5} \leqslant D \leqslant |s|^{1/4}$, and using the convexity bound for $D \geqslant |s|^{1/4}$.

It is now an easy matter to prove Theorem 1. Let N, q and s be as in Theorem 1. We combine Theorems 2 and 3, the latter with $g := f \otimes \chi$ and its conductor $D \mid [N, q^2]$. For convenience we write

$$D = N_0 q^2, \qquad N_0 \leqslant N,$$

and we distinguish between various cases, depending on the relative size of N_0, q and $|s|$. If

$$q \leqslant |s|^{\frac{3\theta}{40-50\theta}} N_0^{-\frac{1}{2}}, \tag{8.1}$$

then

$$\begin{aligned} L(f\otimes\chi, s) &\ll (N_0|s|q)^\varepsilon N_0^{\frac{9}{20-16\theta}} |s|^{\frac{2-\theta}{5-4\theta}} q^{\frac{9}{10-8\theta}} \\ &\ll (N_0|s|q)^\varepsilon N_0^{\frac{32-25\theta}{160-188\theta}} (|s|q)^{\frac{1}{2}-\frac{4-11\theta}{40-47\theta}} \ll N^{\frac{1}{4}} (|s|q)^{\frac{1}{2}-\frac{1}{13}} \end{aligned} \tag{8.2}$$

by Theorem 3, (8.1), and (2.4). If

$$|s|^{\frac{3\theta}{40-50\theta}} N_0^{-\frac{1}{2}} \leqslant q \leqslant |s|^{\frac{1}{10}} N_0^{-\frac{1}{2}}, \tag{8.3}$$

then

$$L(f\otimes\chi, s) \ll (N_0|s|q)^\varepsilon N_0^{\frac{1}{2}} |s|^{\frac{2}{5}} q \ll (N|s|q)^\varepsilon N^{\frac{1}{4}} (|s|q)^{\frac{1}{2}-\frac{1}{22}} \tag{8.4}$$

by Theorem 3 and (8.3). If

$$|s|^{\frac{1}{10}} N_0^{-\frac{1}{2}} \leqslant q \leqslant |s|^{\frac{1}{9}} N_0^{-\frac{2}{9}}, \tag{8.5}$$

then

$$L(f\otimes\chi, s) \ll (N_0|s|q)^\varepsilon N_0^{\frac{5}{4}} |s|^{\frac{1}{4}} q^{\frac{5}{2}} \ll (N|s|q)^\varepsilon N^{\frac{4}{5}} (|s|q)^{\frac{1}{2}-\frac{1}{40}} \tag{8.6}$$

by Theorem 3 and (8.5). If

$$|s|^{\frac{1}{9}} N_0^{-\frac{2}{9}} \leqslant q \leqslant |s|^2, \tag{8.7}$$

then

$$L(f\otimes\chi, s) \ll (N_0|s|q)^\varepsilon N_0^{\frac{3}{4}} |s|^{\frac{1}{2}} q^{\frac{1}{4}} \ll (N|s|q)^\varepsilon N^{\frac{4}{5}} (|s|q)^{\frac{1}{2}-\frac{1}{40}} \tag{8.8}$$

by Theorem 2 and (8.7). If finally

$$q \geqslant |s|^2,$$

then

$$L(f\otimes\chi, s) \ll (N|s|q)^\varepsilon N^{\frac{3}{4}} |s|^{\frac{1}{4}} q^{\frac{3}{8}} \ll (N|s|q)^\varepsilon N^{\frac{3}{4}} (|s|q)^{\frac{1}{2}-\frac{1}{8}} \tag{8.9}$$

by Theorem 2. Here all implied constants depend only on ε and the archimedean parameter μ of f. Theorem 1 now follows from (8.2), (8.4), (8.6), (8.8), and (8.9).

9. Proof of Corollary 2

The proof of Corollary 2 follows along the lines of Appendix 2 in [BHM1]. We indicate some small improvements and extensions to cover all indices n regardless of their square part.

Let us first note that Theorem 7 in [BHM1, Appendix 2] holds for all integers D of the form $D't^2$ where D' is square-free and $t \mid (2M)^\infty$. Now, in the line following [BHM1, Lemma 7.3] we obviously have $|D|_p = |(D,(2M)^\infty)|_p$, since $p \mid 2M$. This and Theorem 7 are the only places in [BHM1, Appendix 2] where it was used that D was assumed to be square-free. Thus Lemma 7.4 reads for integers $D = D't^2$ as above

$$E_p(\varphi_p, \tilde{\varphi}_p, \psi_p, D) \leqslant \frac{2}{|2|_p}(1+p^{-1})(1+p^{1/2})^2(1-p^{-1})^{-3}|(D,(2M)^\infty)|_p^{-1}.$$

With these adjustments, [BHM1, (7.15)] reads

$$\sqrt{D}\rho_f(D) \ll_\varepsilon (kMD)^\varepsilon \left(\Gamma\left(k+\frac{1}{2}\right)\right)^{-1/2} (D,(2M)^\infty)^{1/2} L\left(g\otimes\chi_{D'},\frac{1}{2}\right)^{1/2}, \tag{9.1}$$

valid for special cusp forms $f \in S'_{k+\frac{1}{2}}(4M,\chi)$ as described in the beginning of [BHM1, Section 7.1] and for integers $D = D't^2$ as above. We use Theorem 2 to bound the L-function and we also note [BHM1, Lemma 7.1], that is, we apply Theorem 2 with N replaced by N^2. By Shimura's correspondence, applied to square factors coprime with $2M$, (9.1) holds for all integers $D = D't^2$ without restriction on t. Finally, at the cost of a factor $(\dim S'_{k+\frac{1}{2}}(4M,\chi))^{1/2} \ll_\varepsilon k^{1/2}M^{1/2+\varepsilon}$, we extend the estimate to arbitrary cusp forms $f \in S'_{k+\frac{1}{2}}(4M,\chi)$.

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