AN INTERESTING INDEFINITE INTEGRAL QUADRATIC FORM

GERGELY HARCOS

This note is based on Thomas Browning's MathOverflow post [1].

Theorem. *Let a,b,c be positive integers, and consider the quadratic form*

$$P(x,y) := ax^2 - abcxy - cy^2.$$

The smallest positive value of $P(x,y)$ over $x,y \in \mathbb{Z}$ equals a.

Proof. Let $x_0, y_0 \in \mathbb{Z}$ be such that $P(x_0,y_0) > 0$. We shall show that $P(x_0,y_0) \geqslant a$. The matrix

$$T := \begin{pmatrix} ab^2c+1 & bc \\ ab & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

fixes both the lattice $\mathbb{Z}^2$ and the quadratic form $P(x,y)$. Hence $P(x,y)$ is constant along the orbit of lattice points

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} := T^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \qquad n \in \mathbb{Z}.$$

Note that $P(x_n,y_n) = P(x_0,y_0) > 0$, hence $x_n \neq 0$. Note also that $y_{n+1} = abx_n + y_n$. We shall prove below that there exist both negative and positive values among the y_n's. Hence there exists $n \in \mathbb{Z}$ such that $y_n y_{n+1} \leqslant 0$. However, for such an $n \in \mathbb{Z}$ we have

$$P(x_n,y_n) = ax_n^2 - cy_n y_{n+1} \geqslant ax_n^2 \geqslant a.$$

Hence $P(x_0,y_0) \geqslant a$ as needed.

Now we prove that there exist both negative and positive values among the y_n's. The trace of T exceeds 2, hence T has an eigenvalue $\lambda_1 \in (1,\infty)$ and an eigenvalue $\lambda_2 \in (0,1)$. Diagonalizing T, a straightforward calculation gives that if $n \to \infty$, then

$$(\lambda_1 - \lambda_2)\lambda_1^{-n} T^n \to T - \lambda_2 I \quad \text{and} \quad (\lambda_2 - \lambda_1)\lambda_1^{-n} T^{-n} \to T - \lambda_1 I.$$

In particular,

$$(\lambda_1 - \lambda_2)\lambda_1^{-n} y_n \to abx_0 + (1-\lambda_2)y_0 \quad \text{and} \quad (\lambda_2 - \lambda_1)\lambda_1^{-n} y_{-n} \to abx_0 + (1-\lambda_1)y_0.$$

Multiplying these two limits, another straightforward calculation gives that

$$-(\lambda_1 - \lambda_2)^2 \lambda_1^{-2n} y_n y_{-n} \to ab^2 P(x_0,y_0), \qquad n \to \infty.$$

As $P(x_0,y_0) > 0$, it follows that $y_n y_{-n} < 0$ for every sufficiently large positive integer n. □

REFERENCES

[1] T. Browning, Response to MathOverflow question No. 466399, https://mathoverflow.net/questions/466399

ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS, POB 127, BUDAPEST H-1364, HUNGARY
Email address: gharcos@renyi.hu