THE ADDITIVE AND MULTIPLICATIVE LARGE SIEVE INEQUALITY

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We present classical versions of the additive and multiplicative large sieve inequality, based on the treatment of Iwaniec–Kowalski [3, Section 7.4] and Tao [4].

Theorem 1 (Davenport–Halberstam [2])**.** *Let $\delta > 0$, $M \in \mathbb{Z}$, $N \in \mathbb{N}$. Then for any set of δ-spaced points $x_1, \dots, x_R \in \mathbb{R}/\mathbb{Z}$ and any complex numbers $a_{M+1}, \dots, a_{M+N} \in \mathbb{C}$ we have*

$$\sum_{r=1}^{R} \left| \sum_{n=M+1}^{M+N} a_n e(x_r n) \right|^2 \ll (N + \delta^{-1}) \sum_{n=M+1}^{M+N} |a_n|^2. \tag{1}$$

Theorem 2 (Bombieri–Davenport [1])**.** *Let $Q \in \mathbb{N}$, $M \in \mathbb{Z}$, $N \in \mathbb{N}$. Then for any complex numbers $a_{M+1}, \dots, a_{M+N} \in \mathbb{C}$ we have*

$$\sum_{q=1}^{Q} \sideset{}{^*}\sum_{\chi \bmod q} \frac{q}{\phi(q)} \left| \sum_{n=M+1}^{M+N} a_n \chi(n) \right|^2 \ll (N + Q^2) \sum_{n=M+1}^{M+N} |a_n|^2. \tag{2}$$

Proof of Theorem 1. By duality, the inequality (1) is equivalent to

$$\sum_{n=M+1}^{M+N} \left| \sum_{r=1}^{R} b_r e(x_r n) \right|^2 \ll (N + \delta^{-1}) \sum_{r=1}^{R} |b_r|^2,$$

for arbitrary complex numbers $b_r \in \mathbb{C}$. Let us consider $f(x) := \psi((x - M)/\tilde{N})$, where $\tilde{N} := \max(N, \delta^{-1})$, and $\psi : \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ is a fixed Schwartz class function such that $\psi \geqslant 1$ on $[0, 1]$ and the Fourier transform $\hat{\psi}$ is supported on $(-1, 1)$. Then

$$f \geqslant 1 \text{ on } [M, M + \tilde{N}] \qquad \text{and} \qquad \operatorname{supp} \hat{f} \subset (-1/\tilde{N}, 1/\tilde{N}), \tag{3}$$

so it suffices to show that

$$\sum_{n \in \mathbb{Z}} f(n) \left| \sum_{r=1}^{R} b_r e(x_r n) \right|^2 \ll \tilde{N} \sum_{r=1}^{R} |b_r|^2.$$

Expanding the left hand side, this becomes

$$\sum_{r,r'} b_r \overline{b_{r'}} \left(\sum_{n \in \mathbb{Z}} f(n) e\big((x_r - x_{r'}) n\big) \right) \ll \tilde{N} \sum_{r=1}^{R} |b_r|^2.$$

The pairs $r \neq r'$ do not contribute here, because

$$\sum_{n \in \mathbb{Z}} f(n) e\big((x_r - x_{r'}) n\big) = \sum_{m \in \mathbb{Z}} \hat{f}(m + x_r - x_r') = 0$$

by Poisson summation and $\delta \geqslant 1/\tilde{N}$. The remaining contribution of $r = r'$ is

$$\sum_{r=1}^{R} |b_r|^2 \sum_{n \in \mathbb{Z}} f(n) \ll \tilde{N} \sum_{r=1}^{R} |b_r|^2.$$

□

Proof of Theorem 2. With the notation $c_{\chi,n} := \sqrt{\frac{q}{\phi(q)}}\chi(n)$, the inequality (2) becomes

$$\sum_{\chi}\left|\sum_{n=M+1}^{M+N} c_{\chi,n}a_n\right|^2 \ll (N+Q^2)\sum_{n=M+1}^{M+N}|a_n|^2,$$

where χ runs through all primitive characters with conductor $q \leqslant Q$. By duality, this is equivalent to

$$\sum_{n=M+1}^{M+N}\left|\sum_{\chi} c_{\chi,n}b_\chi\right|^2 \ll (N+Q^2)\sum_{\chi}|b_\chi|^2,$$

for arbitrary complex numbers $b_\chi \in \mathbb{C}$. Let us consider $f(x) := \psi((x-M)/\tilde{N})$, where $\tilde{N} := \max(N, Q^2)$, and $\psi : \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ is as in the previous proof. Then (3) holds, so it suffices to show that

$$\sum_{n\in\mathbb{Z}} f(n)\left|\sum_{\chi} c_{\chi,n}b_\chi\right|^2 \ll \tilde{N}\sum_{\chi}|b_\chi|^2.$$

Expanding the left hand side and using the definition of $c_{\chi,n}$, this becomes

$$\sum_{\chi,\chi'}\sqrt{\frac{qq'}{\phi(q)\phi(q')}}b_\chi\overline{b_{\chi'}}\left(\sum_{n\in\mathbb{Z}} f(n)\chi\overline{\chi'}(n)\right) \ll \tilde{N}\sum_{\chi}|b_\chi|^2.$$

For $\chi \neq \chi'$ the function $n \mapsto \chi\overline{\chi'}(n)$ is periodic by qq' with mean 0, hence we can write it as a linear combination of additive characters $n \mapsto e_{qq'}(rn)$ with $r \not\equiv 0 \pmod{qq'}$. Now Poisson summation and $1/(qq') \geqslant Q^{-2} \geqslant 1/\tilde{N}$ show that

$$\sum_{n\in\mathbb{Z}} f(n)e_{qq'}(rn) = \sum_{m\in\mathbb{Z}}\hat{f}\left(m+\frac{r}{qq'}\right) = 0,$$

therefore the contribution of $\chi \neq \chi'$ is zero. The remaining contribution of $\chi = \chi'$ is

$$\sum_{\chi}\frac{q}{\phi(q)}|b_\chi|^2 \sum_{\substack{n\in\mathbb{Z}\\(n,q)=1}} f(n) \ll \tilde{N}\sum_{\chi}|b_\chi|^2.$$

□

References

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