THE LEE–YANG CIRCLE THEOREM

GERGELY HARCOS

We present a proof of the celebrated Lee–Yang circle theorem [1] based on Ruelle's paper [2]. For an $n\times n$ Hermitian matrix $A=(a_{ij})$ we shall write

$$f_A(z_1,\dots,z_n):=\sum_{S\subset[n]}\Bigg(\prod_{\substack{i\in S\\ j\notin S}}a_{ij}\Bigg)\Bigg(\prod_{k\in S}z_k\Bigg).$$

Theorem (Lee–Yang [1])**.** *Assume that* $|a_{ij}|\leqslant 1$ *and* $|z_k|<1$ *for every* $i,j,k\in[n]$. *Then*

$$f_A(z_1,\dots,z_n)\neq 0.$$

Proof. We proceed by induction on n, treating the a_{ij}'s and the z_k's as complex variables.

For $n=1$ the statement is clear. So let us assume that $n\geqslant 2$, and the statement is true for $n-1$ in place of n. In the definition of f_A, let us separate the contribution of S with $n\notin S$ from the contribution of S with $n\in S$. Then, using $a_{ji}=\overline{a_{ij}}$, we can decompose $f_A(z_1,\dots,z_n)$ as

$$f_B(a_{1,n}\cdot z_1,\dots,a_{n-1,n}\cdot z_{n-1})\ +\ (z_1\cdots z_n)\,\overline{f_B}(a_{1,n}/\overline{z_1},\dots,a_{n-1,n}/\overline{z_{n-1}}),$$

where B is the upper $(n-1)\times(n-1)$ block of A. By the induction hypothesis, the first term is nonzero, hence it suffices to show that

$$\sup_{|a_{i,n}|\leqslant 1}\ \sup_{|z_k|<1}\left|\frac{(z_1\cdots z_{n-1})\,f_B(a_{1,n}/\overline{z_1},\dots,a_{n-1,n}/\overline{z_{n-1}})}{f_B(a_{1,n}\cdot z_1,\dots,a_{n-1,n}\cdot z_{n-1})}\right|\leqslant 1.$$

By continuity, the left-hand side does not change if we strengthen the condition $|a_{i,n}|\leqslant 1$ to $|a_{i,n}|<1$. Then, by the maximum modulus principle, the left-hand side does not change if we replace the condition $|z_k|<1$ by $|z_k|=1$. However, with these new conditions the expression under the supremum is constant 1, and we are done. □

Corollary. *Assume that* $|a_{ij}|\leqslant 1$. *Then the roots of* $f_A(z,\dots,z)$ *lie on the unit circle.*

Proof. We need to show that for $|z|\neq 1$ we have $f_A(z,\dots,z)\neq 0$. For $|z|<1$ the result follows from the above Theorem. For $|z|>1$ it also follows from the above Theorem upon noting that

$$f_A(z,\dots,z)=z^n\,\overline{f_A}(1/\bar z,\dots,1/\bar z).$$ □

References

[1] T. D. Lee, C. N. Yang, *Statistical theory of equations of state and phase transitions. II. Lattice gas and Ising model*, Phys. Rev. **87** (1952), 410–419.

[2] D. Ruelle, *Characterization of Lee-Yang polynomials*, Ann. of Math. **171** (2010), 589–603.

Alfréd Rényi Institute of Mathematics, POB 127, Budapest H-1364, Hungary
Email address: gharcos@renyi.hu