NOTE ON FULLY CHARACTERISTIC SUBMODULES OVER PRINCIPAL IDEAL DOMAINS

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Abstract. We give a description of the fully characteristic submodules of a finitely generated torsion module over a principal ideal domain.

1. Introduction

For a finite group G it is sometimes of interest to determine the fully characteristic subgroups of G, that is, those subgroups which are mapped into themselves under all endomorphisms of G. For example, the commutator subgroup H of G is always of this kind. We have $H \neq G$ whenever G is solvable and $H \neq 1$ whenever G is non-commutative. While it is hopeless to find the fully characteristic subgroups of a finite group G in general, even in the non-solvable case, one can expect a complete description in the commutative case.

Such a description will now be given in a slightly more general setting, namely for a finitely generated torsion module M over a principal ideal domain R. A submodule N of M is called fully characteristic if $\varphi(N) \subseteq N$ holds for all endomorphisms φ of M. In other words, the fully characteristic submodules of M are exactly the submodules of the (R, E)-bimodule ${}_R M_E$ where $E = \operatorname{End}_R(M)$. With these notations we shall prove

Theorem. *Write M as a direct sum of cyclic submodules*

$$M = M_1 \dotplus \cdots \dotplus M_n \tag{1}$$

with exponents satisfying

$$\operatorname{ann} M_1 \mid \cdots \mid \operatorname{ann} M_n. \tag{2}$$

Then the fully characteristic submodules of M are the ones given by

$$N = N_1 \dotplus \cdots \dotplus N_n \tag{3}$$

with $N_i \leq M_i$ $(i = 1, \dots, n)$ and exponents satisfying

$$\operatorname{ann} N_1 \mid \cdots \mid \operatorname{ann} N_n \quad \text{and} \quad \frac{\operatorname{ann} M_1}{\operatorname{ann} N_1} \mid \cdots \mid \frac{\operatorname{ann} M_n}{\operatorname{ann} N_n}. \tag{4}$$

In particular, this set of submodules is independent of the decomposition (1).

2. Proof of the theorem

First of all we observe that it suffices to prove the statement in the case when R is a local ring. To see this consider the prime ideal decomposition of the exponent of M

$$\operatorname{ann} M = (\pi_1)^{e_1} \cdots (\pi_g)^{e_g}.$$

For any R-module L and any $t \in L$ let $L[t]$ be the set of those elements of L which are annihilated by t. Then

$$M = M[\pi_1^{e_1}] \dotplus \cdots \dotplus M[\pi_g^{e_g}]$$

where each component $M[\pi_i^{e_i}]$ is isomorphic to an R_{π_i}-module with R_{π_i} the localization of R at the prime ideal (π_i). Similarly, any submodule N of M can be written as

$$N = N[\pi_1^{e_1}] \dotplus \cdots \dotplus N[\pi_g^{e_g}]$$

where of course the component $N[\pi_i^{e_i}]$ is a submodule of $M[\pi_i^{e_i}]$. The decompositions of M and N into n terms given in the theorem correspond bijectively to a set of similar decompositions of the components $M[\pi_i^{e_i}]$ and $N[\pi_i^{e_i}]$ into n terms. Finally, it is also clear from the definition of these components that N is fully characteristic in M if and only if each $N[\pi_i^{e_i}]$ is fully characteristic in $M[\pi_i^{e_i}]$.

Hence we assume that R is a local ring with the unique prime ideal (π) and M is an R-module with the decomposition (1) as a direct sum of cyclic submodules M_i and exponents satisfying (2). Let N be a fully characteristic submodule of M. For any $i = 1, \ldots, n$ let N_i be the image of N under the projection of M onto its ith component M_i in (1). Since this projection is an endomorphism of M and N is fully characteristic in M, we get $N_i \leq N$. Therefore

$$N_1 \dotplus \cdots \dotplus N_n \leq N.$$

On the other hand, it is obvious without any assumption on the submodule N of M that

$$N \leq N_1 \dotplus \cdots \dotplus N_n.$$

Therefore (3) holds with the submodules $N_i \leq M_i$ $(i = 1, \ldots, n)$.

Now take any submodule N in M of the form (3) with $N_i \leq M_i$ $(i = 1, \ldots, n)$. Introduce the notations $\operatorname{ann} M_i = (\pi)^{e_i}$ and $\operatorname{ann} M_i/N_i = (\pi)^{f_i}$ for $i = 1, \ldots, n$. Then (2) can be written as

$$e_1 \leq \cdots \leq e_n \tag{5}$$

and we need to prove that N is fully characteristic in M if and only if (4) holds. Clearly, (4) is the same as

$$e_1 - f_1 \leq \cdots \leq e_n - f_n \quad \text{and} \quad f_1 \leq \cdots \leq f_n. \tag{6}$$

Choose a generator m_i for each of the cyclic components M_i ($i = 1, \dots, n$) and take any endomorphism φ of M. Such an endomorphism can be described by an $n \times n$ matrix $(a_{ij}) \in M_n(R)$ satisfying

$$\varphi(m_i) = \sum_{j=1}^{n} a_{ij} m_j. \tag{7}$$

If we write $a_{ij} = \pi^{e_{ij}} b_{ij}$ where $b_{ij} \in R^\times \cup \{0\}$ then

$$0 = \varphi(0) = \varphi(\pi^{e_i} m_i) = \pi^{e_i} \sum_{j=1}^{n} a_{ij} m_j = \sum_{j=1}^{n} \pi^{e_i + e_{ij}} b_{ij} m_j$$

shows that $\pi^{e_i + e_{ij}} b_{ij} m_j = 0$ for all $i, j \in \{1, \dots, n\}$, that is,

$$b_{ij} \neq 0 \Longrightarrow e_j - e_i \leq e_{ij} \quad (1 \leq i, j \leq n). \tag{8}$$

Conversely, if this condition holds on the matrix (a_{ij}) then it determines uniquely an endomorphism φ of M via (7). If we observe that for each $i = 1, \dots, n$ the element $n_i = \pi^{f_i}$ generates the module N_i, it becomes clear that φ maps N into itself if and only if $\varphi(n_i)$ lies in N for all i. But then

$$\varphi(n_i) = \varphi(\pi^{f_i} m_i) = \pi^{f_i} \sum_{j=1}^{n} a_{ij} m_j = \sum_{j=1}^{n} \pi^{f_i + e_{ij}} b_{ij} m_j$$

shows that $\varphi(N) \subseteq N$ is equivalent to

$$b_{ij} \neq 0 \Longrightarrow f_j - f_i \leq e_{ij} \quad (1 \leq i, j \leq n). \tag{9}$$

We can summarize our result as follows. N is fully characteristic in M if and only if (8) implies (9) for any choice of the elements $b_{ij} \in R^\times \cup \{0\}$ and the nonnegative integers e_{ij}. By choosing $b_{ij} = 1$ and $e_{ij} = \max(0, e_j - e_i)$ we find that

$$f_j - f_i \leq \max(0, e_j - e_i) \quad (1 \leq i, j \leq n)$$

is a necessary condition for N to be fully characteristic in M. Clearly, the condition is sufficient at the same time. By (5), we can rewrite this condition as (6) which completes the proof of the theorem. $\square$