THE PRIME GEODESIC THEOREM IN ARITHMETIC PROGRESSIONS

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Abstract. We address the prime geodesic theorem in arithmetic progressions, and resolve conjectures of Golovchanskiĭ–Smotrov (1999). In particular, we prove that the traces of closed geodesics on the modular surface do not equidistribute in the reduced residue classes of a given modulus.

1. Introduction

1.1. Historical prelude. The prime geodesic theorem asks for an asymptotic evaluation of the counting function of oriented primitive closed geodesics on hyperbolic manifolds. If the underlying group is a cofinite Fuchsian group $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$, then this problem has received distinguished attention among number theorists; see for instance [Hej76a, Hej76b, Hej83, Hub61a, Hub61b, Kuz78, Sar80, Sel14, Ven90] and references therein for classical results.

A closed geodesic P on $\Gamma\backslash\mathbb{H}$ corresponds bijectively to a hyperbolic conjugacy class in Γ (cf. [Hub59]). If $\mathrm{N}(P)$ denotes the norm of this conjugacy class, then the hyperbolic length of P equals $\log \mathrm{N}(P)$. As usual, we set $\Lambda_\Gamma(P) = \log \mathrm{N}(P_0)$, where P_0 is the primitive closed geodesic underlying P, and we introduce the Chebyshev-like counting function

$$\Psi_\Gamma(x) \coloneqq \sum_{\mathrm{N}(P) \leq x} \Lambda_\Gamma(P).$$

Because the prime geodesic theorem is reminiscent of the prime number theorem, the norms are sometimes called pseudoprimes. Selberg [Sel14] established an asymptotic formula of the shape

$$\Psi_\Gamma(x) = \sum_{\frac{1}{2} < s_j \leq 1} \frac{x^{s_j}}{s_j} + \mathcal{E}_\Gamma(x), \tag{1.1}$$

where the main term emerges from the small eigenvalues $\lambda_j = s_j(1-s_j) < \frac{1}{4}$ of the Laplacian on $\Gamma\backslash\mathbb{H}$ for the upper half-plane $\mathbb{H}$, and $\mathcal{E}_\Gamma(x)$ is an error term. It is known that $\mathcal{E}_\Gamma(x) \ll_{\Gamma,\varepsilon} x^{\frac{3}{4}+\varepsilon}$; see the explicit formulæ in [Iwa84a, KK22]. This barrier is often termed the trivial bound. Since an analogue of the Riemann Hypothesis holds for Selberg zeta functions apart from a finite number of exceptional zeros, the analogy with the prime number theorem suggests that the best possible estimate should be $\mathcal{E}_\Gamma(x) \ll_{\Gamma,\varepsilon} x^{\frac{1}{2}+\varepsilon}$. This remains unresolved due to the abundance of Laplace eigenvalues.

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If Γ is arithmetic, then an improvement over the barrier $\frac{3}{4}$ was achieved by Iwaniec [Iwa84a], who proved that $\mathcal{E}_\Gamma(x) \ll_\varepsilon x^{\frac{35}{48}+\varepsilon}$ for the full modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. It was also observed by Iwaniec [Iwa84b, pp. 187–189] that the stronger exponent $\frac{2}{3}+\varepsilon$ follows from the Generalised Lindelöf Hypothesis stated either for classical Rankin–Selberg L-functions in the eigenvalue aspect or for quadratic Dirichlet L-functions in the conductor aspect. Luo–Sarnak [LS95] then strengthened the machinery of Iwaniec to obtain the exponent $\frac{7}{10}+\varepsilon$; see also [Koy98, LRS95]. As a further refinement, Cai [Cai02] derived the exponent $\frac{71}{102}+\varepsilon$. The crucial gist in all these works is to estimate nontrivially a certain spectral exponential sum via the Kuznetsov formula. On the other hand, the subsequent work of Soundararajan–Young [SY13] demonstrated that

$$\mathcal{E}_\Gamma(x) \ll_\varepsilon x^{\frac{2}{3}+\frac{\vartheta}{6}+\varepsilon},$$

where ϑ is a subconvex exponent for quadratic Dirichlet L-functions. The current record $\vartheta = \frac{1}{6}$ by Conrey–Iwaniec [CI00] implies the best known exponent $\frac{25}{36}+\varepsilon$. The proof combines the Kuznetsov–Bykovskiĭ formula (see [Byk94, Kuz78, SY13]) with the Selberg trace formula (see [Hej78, Hej83]). For recent progress on the prime geodesic theorem and its generalisations, we direct the reader to [BBCL22, BBHM19, BCC$^+$19, BF19, BF20, BFR22, Bir18, Bir24, CCL22, CKS23, CG18, CWZ22, DM23, Has13, Kan20, Kan22, Kan23, Kan24a, Kan24b, PR17].

1.2. Statement of main results. By a classical theorem of Dirichlet, the primes equidistribute in the reduced residue classes of a given modulus. As we shall see, the prime geodesic analogue of this phenomenon breaks down, and the corresponding non-uniform distribution can be determined explicitly.

We fix $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. By definition, an element $P \in \Gamma$ is said to be hyperbolic if as a Möbius transformation

$$Pz = \frac{az+b}{cz+d},$$

it possesses two distinct real fixed points. By conjugation, any hyperbolic element $P \in \Gamma$ may be expressed as $P = \sigma^{-1}\widetilde{P}\sigma$, where $\sigma \in \mathrm{PSL}_2(\mathbb{R})$ and $\widetilde{P} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ with $\lambda > 1$. Here $\widetilde{P}$ acts as multiplication by λ^2, namely $\widetilde{P}z = \lambda^2 z$. The factor λ^2 is called the norm of P, which depends only on the $\mathrm{PSL}_2(\mathbb{R})$-conjugacy class of P. We note that the positive trace

$$t = \mathrm{tr}(P) = \mathrm{tr}(\widetilde{P}) = \lambda + \lambda^{-1}$$

is an integer exceeding 2, and hence the norm takes the form

$$\mathrm{N}(P) = \left(\frac{t+\sqrt{t^2-4}}{2}\right)^2 = t^2 - 2 + O(t^{-2}).$$

Given a prime $p \geq 2$, we define the Chebyshev-like counting function in arithmetic progressions modulo p by

$$\Psi_\Gamma(x; p, a) \coloneqq \sum_{\substack{\mathrm{N}(P) \leq x \\ \mathrm{tr}(P) \equiv a \, (\mathrm{mod}\, p)}} \Lambda_\Gamma(P).$$

The following result shows that the main term in an asymptotic for $\Psi_\Gamma(x; p, a)$ depends on the residue class $a \, (\mathrm{mod}\, p)$ unlike for primes in arithmetic progressions.

Theorem 1.1. *Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ be the modular group. Moreover, let $p \geq 3$ be a prime number, and let $a \pmod p$ be an arbitrary residue class. Then we have that*

$$\Psi_\Gamma(x; p, a) = \begin{cases} \dfrac{1}{p-1} \cdot x + O_\varepsilon(x^{\frac{3}{4}+\frac{\vartheta}{2}+\varepsilon}) & \text{if } \left(\dfrac{a^2-4}{p}\right) = 1, \\ \dfrac{1}{p+1} \cdot x + O_\varepsilon(x^{\frac{3}{4}+\frac{\vartheta}{2}+\varepsilon}) & \text{if } \left(\dfrac{a^2-4}{p}\right) = -1, \\ \dfrac{p}{p^2-1} \cdot x + O_{p,\varepsilon}(x^{\frac{3}{4}+\frac{\vartheta}{2}+\varepsilon}) & \text{if } \left(\dfrac{a^2-4}{p}\right) = 0, \end{cases} \tag{1.2}$$

where ϑ is a subconvex exponent for quadratic Dirichlet L-functions.

Remark 1. The implied constant in the error term is independent of p when $a \not\equiv \pm 2 \pmod p$.

Remark 2. When $p = 3$, the first case of (1.2) is void, while the second case is covered with a stronger error term by [GS99, Theorem 1].

Remark 3. Apart from the size of the error term, Theorem 1.1 resolves [GS99, Conjecture 2] for level $N = 1$. In fact, we expect that our method works for $\Gamma = \Gamma_0(N)$ when $(N, p) = 1$, but we restrict to $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ for simplicity. Furthermore, we expect that the error term can be improved significantly by a more careful analysis (e.g. by combining the Kuznetsov–Bykovskiĭ formula with an adèlic trace formula), but we solely focused on determining the main term according to the sign of the Legendre symbol. We leave such pursuits for future work.

The method of Golovchanskiĭ–Smotrov [GS99, Theorem 1] is different from ours, and they delve into properties of the norms and traces, expressing $\Psi_\Gamma(x; p, a)$ as a linear combination of $\Psi_{\Gamma_0(2^k)}(x)$ for some $k \geq 0$, for which an asymptotic formula is already known as in (1.1). For example, they derived a general linear combination of the shape

$$3\Psi_{\Gamma_0(N)}(x) - 3\Psi_{\Gamma_0(2N)}(x) + \Psi_{\Gamma_0(4N)}(x) = 3\Psi_{\Gamma_0(N)}(x; 2, 1),$$

from which it follows that[1]

$$\Psi_{\Gamma_0(N)}(x; 2, 1) = \frac{1}{3} \cdot x + \mathcal{E}_{\Gamma_0(N)}(x), \qquad \Psi_{\Gamma_0(N)}(x; 2, 0) = \frac{2}{3} \cdot x + \mathcal{E}_{\Gamma_0(N)}(x).$$

The level structure that they developed is delicate, and it appears that their idea only works for some specific values of p and a. Hence, some new ideas are needed to prove Theorem 1.1.

An elementary counting argument implies the following result.

[1] We emphasise that the case of $p = 2$ is not contained in Theorem 1.1.

Corollary 1.2. *Let* $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$*, and let* $p \geq 3$ *be a prime. Then we have that*

$$\sum_{\substack{a \,(\mathrm{mod}\, p) \\ \left(\frac{a^2-4}{p}\right)=1}} \Psi_\Gamma(x; p, a) = \frac{p-3}{2(p-1)} \cdot x + O_{p,\varepsilon}(x^{\frac{3}{4}+\frac{\vartheta}{2}+\varepsilon}),$$

$$\sum_{\substack{a \,(\mathrm{mod}\, p) \\ \left(\frac{a^2-4}{p}\right)=-1}} \Psi_\Gamma(x; p, a) = \frac{p-1}{2(p+1)} \cdot x + O_{p,\varepsilon}(x^{\frac{3}{4}+\frac{\vartheta}{2}+\varepsilon}),$$

$$\sum_{\substack{a \,(\mathrm{mod}\, p) \\ \left(\frac{a^2-4}{p}\right)=0}} \Psi_\Gamma(x; p, a) = \frac{2p}{p^2-1} \cdot x + O_{p,\varepsilon}(x^{\frac{3}{4}+\frac{\vartheta}{2}+\varepsilon}),$$

where ϑ *is a subconvex exponent for quadratic Dirichlet* L*-functions.*

Remark 4. Apart from the size of the error term, Corollary 1.2 resolves [GS99, Conjecture 1] in the case of full level $N = 1$, and again the method should work for $\Gamma = \Gamma_0(N)$ when $(N, p) = 1$.

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2. Key propositions

This section prepares for the proof of Theorem 1.1. Throughout, we follow [SY13, Section 2] closely.

Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ as before. Sarnak [Sar82, Proposition 1.4] showed that the primitive hyperbolic conjugacy classes in Γ correspond bijectively to the Γ-equivalence classes of primitive indefinite binary quadratic forms. We recall this correspondence briefly. For a given primitive quadratic form $ax^2 + bxy + cy^2$ of discriminant $d > 0$, the automorphs are the elements

$$P(t, u) = \begin{pmatrix} \frac{t - bu}{2} & -cu \\ au & \frac{t + bu}{2} \end{pmatrix} \in \Gamma,$$

with $t^2 - du^2 = 4$ being a solution of the Pell equation. For u nonzero, $P(t, u)$ is hyperbolic with norm $(t + u\sqrt{d})^2/4$ and trace t. Because $P(-t, -u) = P(t, u)$ holds in Γ, we shall restrict to $t > 0$ without loss of generality. This is in harmony with our convention in Section 1.2 that $\mathrm{tr}(P) > 2$ for a hyperbolic element $P \in \Gamma$. Let (t_d, u_d) be the fundamental solution of the Pell equation, and let ε_d be the corresponding fundamental unit (of the quadratic order of discriminant d):

$$\varepsilon_d := \frac{t_d + \sqrt{d} u_d}{2}.$$

Then $P(t_d, u_d)$ is a primitive hyperbolic matrix of norm ε_d^2 and trace t_d. Moreover, every automorph $P(t, u)$ with $u > 0$ (resp. $u < 0$) is a unique positive (resp. negative) integral power of $P(t_d, u_d)$. Sarnak's bijection sends the quadratic form $ax^2 + bxy + cy^2$ to the conjugacy class of $P(t_d, u_d)$ in Γ. Thus, for a given discriminant $d > 0$, there are $h(d)$

primitive hyperbolic conjugacy classes in Γ (the class number of d), each of norm ε_d^2 and trace t_d.

Now every hyperbolic conjugacy class $\{P\}$ can be written uniquely as $\{P_0^n\}$ for $n \geq 1$ and a primitive hyperbolic conjugacy class $\{P_0\}$ (cf. [Hub59]). Combining this with Sarnak's bijection described above, we obtain

$$\Psi_\Gamma(x;p,a) = 2 \sum_{\substack{3 \leq t \leq X \\ t \equiv a \,(\mathrm{mod}\, p)}} \sum_{t^2-du^2=4} h(d) \log \varepsilon_d,$$

where X abbreviates $\sqrt{x} + \frac{1}{\sqrt{x}}$, and $d > 0$ (resp. $u > 0$) runs through discriminants (resp. integers). The class number formula $h(d)\log\varepsilon_d = \sqrt{d}L(1,\chi_d)$, where χ_d is the not necessarily primitive quadratic Dirichlet character associated to the discriminant d, allows us to write

$$\Psi_\Gamma(x;p,a) = 2 \sum_{\substack{3 \leq t \leq X \\ t \equiv a \,(\mathrm{mod}\, p)}} \sum_{t^2-du^2=4} \sqrt{d}L(1,\chi_d). \tag{2.1}$$

For an arbitrary discriminant $\delta > 0$, we define Zagier's L-series by (cf. [SY13, (6) & (3)])

$$L(s,\delta) \coloneqq \sum_{du^2=\delta} L(s,\chi_d)u^{1-2s} = \sum_{q=1}^{\infty} \frac{\lambda_q(\delta)}{q^s},$$

where $d > 0$ (resp. $u > 0$) runs through discriminants (resp. integers). This series admits a transparent Euler product expansion (to be discussed below), while it simplifies (2.1) as

$$\Psi_\Gamma(x;p,a) = 2 \sum_{\substack{3 \leq t \leq X \\ t \equiv a \,(\mathrm{mod}\, p)}} \sqrt{t^2-4}L(1,t^2-4). \tag{2.2}$$

If $\delta = Dl^2$, where $D > 0$ is a fundamental discriminant and $l > 0$ is an integer, then we obtain the following Euler product expansion of Zagier's L-series (cf. [SY13, (2)]):

$$\begin{aligned} L(s,\delta) &= \sum_{u \mid l} L(s,\chi_{Dl^2/u^2})u^{1-2s} \\ &= L(s,\chi_D) \sum_{u \mid l} u^{1-2s} \prod_{\boldsymbol{p} \mid \frac{l}{u}} (1-\chi_D(\boldsymbol{p})) \\ &= \prod_{\boldsymbol{p}} \left(\sum_{0 \leq m < v_{\boldsymbol{p}}(l)} \boldsymbol{p}^{m(1-2s)} + \frac{\boldsymbol{p}^{v_{\boldsymbol{p}}(l)(1-2s)}}{1-\chi_D(\boldsymbol{p})\boldsymbol{p}^{-s}} \right). \end{aligned} \tag{2.3}$$

In particular, for fixed δ, the arithmetic function $q \mapsto \lambda_q(\delta)$ is multiplicative. The idea of the proof of Theorem 1.1 is to group together certain values of t in (2.2) such that the corresponding Zagier L-series $L(s,t^2-4)$ has a constant Euler factor at $\boldsymbol{p} = p$. Thus we are led to consider $L(s,\delta)$ with its Euler factor at $\boldsymbol{p} = p$ removed:

$$L^p(s,\delta) \coloneqq \sum_{\substack{q \geq 1 \\ (q,p)=1}} \frac{\lambda_q(\delta)}{q^s}.$$

As we shall see, for $t \equiv a \,(\mathrm{mod}\, p)$ the Euler factor at $\boldsymbol{p} = p$ in $L(s,t^2-4)$ is constant when $a \not\equiv \pm 2 \,(\mathrm{mod}\, p)$, but it depends on $v_p(t-a)$ and other quantities when $a \equiv \pm 2 \,(\mathrm{mod}\, p)$.

This motivates to consider higher congruences $t \equiv r \pmod{p^n}$, as in the two key propositions below.

Proposition 2.1. *Let $p \geq 3$ be a prime, and let $n \geq 1$ be an integer. Let $r \pmod{p^n}$ be an arbitrary residue class. If $(q,p) = 1$ and b denotes the squarefree part of q, then*

$$\sum_{\substack{3 \leq t \leq X \\ t \equiv r \pmod{p^n}}} \lambda_q(t^2-4) = \frac{X}{p^n} \cdot \frac{\mu(b)}{b} + O_\varepsilon(q^{\frac{1}{2}+\varepsilon}).$$

Proof. It follows from [SY13, Lemma 2.3] that

$$\lambda_q(t^2-4) = \sum_{q_1^2 q_2 = q} \frac{1}{q_2} \sum_{k \pmod{q_2}} e\left(\frac{kt}{q_2}\right) S(k^2,1;q_2).$$

This leads to

$$\sum_{\substack{3 \leq t \leq X \\ t \equiv r \pmod{p^n}}} \lambda_q(t^2-4) = \sum_{q_1^2 q_2 = q} \frac{1}{q_2} \sum_{k \pmod{q_2}} S(k^2,1;q_2) \sum_{\substack{3 \leq t \leq X \\ t \equiv r \pmod{p^n}}} e\left(\frac{kt}{q_2}\right).$$

For $k \equiv 0 \pmod{q_2}$, the inner sum over t is $\frac{X}{p^n} + O(1)$, and $S(0,1;q_2) = \mu(q_2)$, yielding the expected main term. We estimate the contribution of $k \not\equiv 0 \pmod{q_2}$ via Weil's bound for Kloosterman sums and the bound

$$\sum_{k \not\equiv 0 \pmod{q_2}} \left| \sum_{\substack{3 \leq t \leq X \\ t \equiv r \pmod{p^n}}} e\left(\frac{kt}{q_2}\right) \right| \ll \sum_{k \not\equiv 0 \pmod{q_2}} \left\| \frac{kp^n}{q_2} \right\|^{-1} \ll q_2 \log q_2,$$

where $\|\cdot\|$ is the distance to the nearest integer. In the last step, we used that the map $k \mapsto kp^n$ permutes the nonzero residue classes modulo q_2, since $(q_2,p) = 1$. The proof is complete. □

Guided by Proposition 2.1 and (2.2), we consider the sum

$$\Psi_\Gamma^\star(x;p^n,r) \coloneqq 2 \sum_{\substack{3 \leq t \leq X \\ t \equiv r \pmod{p^n}}} \sqrt{t^2-4}\, L^p(1,t^2-4). \tag{2.4}$$

We shall deduce Theorem 1.1 from the following analogue of [SY13, Theorem 3.2]:

Proposition 2.2. *Let $p \geq 3$ be a prime, and let $n \geq 1$ be an integer. Let $r \pmod{p^n}$ be an arbitrary residue class. Then*

$$\Psi_\Gamma^\star(x+u;p^n,r) - \Psi_\Gamma^\star(x;p^n,r) = \frac{u}{p^n} + O_\varepsilon(u^{\frac{1}{2}} x^{\frac{1}{4}+\frac{\vartheta}{2}+\varepsilon}), \qquad \sqrt{x} \leq u \leq x. \tag{2.5}$$

Proof. We recall that X abbreviates $\sqrt{x} + \frac{1}{\sqrt{x}}$ and now set

$$X' \coloneqq \sqrt{x+u} + \frac{1}{\sqrt{x+u}}, \qquad \sqrt{x} \leq u \leq x.$$

From the definition (2.4), it is clear that

$$\begin{aligned}\Psi_\Gamma^\star(x+u;p^n,r)-\Psi_\Gamma^\star(x;p^n,r) &= 2\sum_{\substack{X<t\leq X'\\ t\equiv r\,(\mathrm{mod}\,p^n)}}\sqrt{t^2-4}L^p(1,t^2-4)\\ &= \left(2+O\left(x^{-1}\right)\right)\sum_{\substack{X<t\leq X'\\ t\equiv r\,(\mathrm{mod}\,p^n)}} tL^p(1,t^2-4),\end{aligned}$$

because $\sqrt{t^2-4}=t(1+O(t^{-2}))$. We shall approximate $L^p(1,t^2-4)$ in terms of a suitable Dirichlet series. Let $V\geq 1$ be a parameter to be chosen later, and let

$$\delta=t^2-4=Dl^2,$$

where $D>0$ is a fundamental discriminant and $l>0$ is an integer. Consider

$$S_V^p(\delta)\coloneqq\sum_{\substack{q\geq 1\\ (q,p)=1}}\frac{\lambda_q(\delta)}{q}e^{-\frac{q}{V}}.$$

Shifting the contour yields the expression

$$S_V^p(\delta)=\int_{(1)}L^p(1+s,\delta)V^s\Gamma(s)\frac{ds}{2\pi i}=L^p(1,\delta)+\int_{(-\frac{1}{2})}L^p(1+s,\delta)V^s\Gamma(s)\frac{ds}{2\pi i}.$$

On the right-hand side, for some $A>0$,

$$L^p(1+s,\delta)\ll|L(1+s,\delta)|\ll_\varepsilon|L(1+s,\chi_D)|l^\varepsilon\ll_\varepsilon\delta^{\vartheta+\varepsilon}|s|^A,\qquad \mathrm{Re}(s)=-\frac{1}{2},$$

while $\Gamma(s)$ decays exponentially. It follows that

$$L^p(1,\delta)=S_V^p(\delta)+O_\varepsilon(\delta^{\vartheta+\varepsilon}V^{-\frac{1}{2}}),$$

and hence

$$\Psi_\Gamma^\star(x+u;p^n,r)-\Psi_\Gamma^\star(x;p^n,r)=(2+O(x^{-1}))\sum_{\substack{X<t\leq X'\\ t\equiv r\,(\mathrm{mod}\,p^n)}}tS_V^p(t^2-4)+O_\varepsilon(ux^{\vartheta+\varepsilon}V^{-\frac{1}{2}}).$$

If $(q,p)=1$ and $q=bc^2$ with b squarefree, then Proposition 2.1 along with partial summation leads to

$$2\sum_{\substack{X<t\leq X'\\ t\equiv r\,(\mathrm{mod}\,p^n)}}t\lambda_q(t^2-4)=\frac{u}{p^n}\cdot\frac{\mu(b)}{b}+O_\varepsilon(Xq^{\frac{1}{2}+\varepsilon}).$$

It thus follows that

$$2\sum_{\substack{X<t\leq X'\\ t\equiv r\,(\mathrm{mod}\,p^n)}}tS_V^p(t^2-4)=\frac{u}{p^n}\sum_{\substack{b,c\geq 1\\ (bc,p)=1}}\frac{\mu(b)}{b^2c^2}e^{-\frac{bc^2}{V}}+O_\varepsilon(XV^{\frac{1}{2}+\varepsilon}).$$

A standard contour shift argument gives

$$\sum_{\substack{b,c\geq 1\\ (bc,p)=1}}\frac{\mu(b)}{b^2c^2}e^{-\frac{bc^2}{V}}=\int_{(1)}V^s\Gamma(s)\frac{\zeta^p(2+2s)}{\zeta^p(2+s)}\frac{ds}{2\pi i}=1+O(V^{-\frac{1}{2}}),$$

where $\zeta^p(s)$ is the Riemann zeta function with the Euler factor at p removed. As a result,

$$\Psi_\Gamma^\star(x+u;p^n,r)-\Psi_\Gamma^\star(x;p^n,r)=\frac{u}{p^n}+O_\varepsilon(x^{\frac12}V^{\frac12+\varepsilon}+ux^{\vartheta+\varepsilon}V^{-\frac12}).$$

Setting $V=ux^{-\frac12+\vartheta}$ yields (2.5). □

Corollary 2.3. *Let $p\geq 3$ be a prime, and let $n\geq 1$ be an integer. Let $r \pmod{p^n}$ be an arbitrary residue class. Then*

$$\Psi_\Gamma^\star(x;p^n,r)=\frac{x}{p^n}+O_\varepsilon(x^{\frac34+\frac{\vartheta}{2}+\varepsilon}). \tag{2.6}$$

Proof. Setting $u=x$ in (2.5) yields

$$\Psi_\Gamma^\star(2x;p^n,r)-\Psi_\Gamma^\star(x;p^n,r)=\frac{x}{p^n}+O_\varepsilon(x^{\frac34+\frac{\vartheta}{2}+\varepsilon}).$$

Now (2.6) follows readily by a dyadic subdivision. □

3. Proof of Theorem 1.1

We shall approximate the t-sum in (2.2). As before, we write

$$\delta=t^2-4=Dl^2,$$

where $D>0$ is a fundamental discriminant and $l>0$ is an integer.

If $a\not\equiv\pm2 \pmod p$, then for every t participating in (2.2), we have that $p\nmid l$ and

$$\chi_D(p)=\left(\frac{D}{p}\right)=\left(\frac{Dl^2}{p}\right)=\left(\frac{t^2-4}{p}\right)=\left(\frac{a^2-4}{p}\right).$$

By (2.3), the corresponding Zagier L-series $L(s,t^2-4)$ factorises as

$$L(s,t^2-4)=\left(1-\left(\frac{a^2-4}{p}\right)p^{-s}\right)^{-1}L^p(s,t^2-4),$$

hence by (2.2) and (2.4), it also follows that

$$\Psi_\Gamma(x;p,a)=\left(1-\left(\frac{a^2-4}{p}\right)p^{-1}\right)^{-1}\Psi_\Gamma^\star(x;p,a).$$

Applying (2.6), we obtain the first two cases of (1.2).

If $a\equiv\pm2 \pmod p$, then we shall assume (without loss of generality) that $a=\pm2$. We subdivide the t-sum in (2.2) according to the exponent of p in the positive integer $t-a$:

$$\Psi_\Gamma(x;p,a)=\sum_{k=1}^{\infty}\Psi_\Gamma(x;p,a;k),$$

where

$$\Psi_\Gamma(x;p,a;k)\coloneqq 2\sum_{\substack{3\leq t\leq X\\ v_p(t-a)=k}}\sqrt{t^2-4}L(1,t^2-4).$$

We shall approximate these pieces individually. Note that $\Psi_\Gamma(x;p,a;k)=0$ for $p^k>X-a$. Moreover, the condition $v_p(t-a)=k$ constrains t to $p-1$ residue classes modulo p^{k+1}, and it also yields $v_p(t^2-4)=k$.

If $k = 2m-1$ is odd, then $p \mid D$ and $v_p(l) = m-1$, hence by (2.3),

$$L(s,t^2-4) = \frac{1-p^{m(1-2s)}}{1-p^{1-2s}} L^p(s,t^2-4).$$

Using also (2.4) and (2.6), we obtain

$$\begin{aligned}\Psi_\Gamma(x;p,a;2m-1) &= \frac{p-1}{p^{2m}} \cdot \frac{1-p^{-m}}{1-p^{-1}} \cdot x + O_{p,\varepsilon}(x^{\frac{3}{4}+\frac{\vartheta}{2}+\varepsilon}) \\ &= (p^{1-2m}-p^{1-3m})x + O_{p,\varepsilon}(x^{\frac{3}{4}+\frac{\vartheta}{2}+\varepsilon}). \end{aligned} \tag{3.1}$$

It is important that the implied constant in the error term is independent of m.

If $k = 2m$ is even, then $p \nmid D$ and $v_p(l) = m$, hence by (2.3),

$$L(s,t^2-4) = \left(\frac{1-p^{m(1-2s)}}{1-p^{1-2s}} + \frac{p^{m(1-2s)}}{1-\chi_D(p)p^{-s}}\right) L^p(s,t^2-4).$$

We can understand $\chi_D(p)$ by writing $t = a + p^{2m}r$. Indeed, then $t^2-4 = 2ap^{2m}r + p^{4m}r^2$, hence

$$\chi_D(p) = \left(\frac{D}{p}\right) = \left(\frac{Dl^2p^{-2m}}{p}\right) = \left(\frac{2ar}{p}\right).$$

This means that among the $p-1$ choices for $t \pmod{p^{2m+1}}$, half the time $\chi_D(p)$ equals 1 and half the time it equals -1. Using also (2.4) and (2.6), we obtain

$$\begin{aligned}\Psi_\Gamma(x;p,a;2m) &= \frac{p-1}{p^{2m+1}}\left(\frac{1-p^{-m}}{1-p^{-1}} + \frac{(1/2)p^{-m}}{1-p^{-1}} + \frac{(1/2)p^{-m}}{1+p^{-1}}\right)x + O_{p,\varepsilon}(x^{\frac{3}{4}+\frac{\vartheta}{2}+\varepsilon}) \\ &= \left(p^{-2m} - \frac{p^{-3m}}{p+1}\right)x + O_{p,\varepsilon}(x^{\frac{3}{4}+\frac{\vartheta}{2}+\varepsilon}). \end{aligned} \tag{3.2}$$

Again, the implied constant in the error term is independent of m.

Summing up the pieces $\Psi_\Gamma(x;p,a;2m-1)$ and $\Psi_\Gamma(x;p,a;2m)$ for $1 \le m \le \log(X+2)$, and inserting the approximations (3.1)–(3.2), we deduce the asymptotic formula

$$\Psi_\Gamma(x;p,\pm 2) = c_p x + O_{p,\varepsilon}(x^{\frac{3}{4}+\frac{\vartheta}{2}+\varepsilon}),$$

where

$$c_p \coloneqq \sum_{m=1}^{\infty}\left(p^{1-2m} - p^{1-3m} + p^{-2m} - \frac{p^{-3m}}{p+1}\right) = \frac{p}{p^2-1}.$$

This is the third case of (1.2), and the proof of Theorem 1.1 is complete.

4. Proof of Corollary 1.2

There are $\frac{p-3}{2}$ residues $a \pmod p$ such that a^2-4 is a nonzero quadratic residue. Indeed, this occurs if and only if $a^2-4 \equiv b^2 \pmod p$ for some $b \not\equiv 0 \pmod p$, which can be written as $(a+b)(a-b) \equiv 4 \pmod p$. Making the change of variables $a+b \equiv 2x \pmod p$ and $a-b \equiv 2x^{-1} \pmod p$, we obtain $a \equiv x + x^{-1} \pmod p$ and $b \equiv x - x^{-1} \pmod p$ with the condition $x \not\equiv -1, 0, 1 \pmod p$. This restriction leaves $\frac{p-3}{2}$ different ways to choose $a \pmod p$. Since there are two solutions to $(\frac{a^2-4}{p}) = 0$, we conclude that there are $\frac{p-1}{2}$ residues $a \pmod p$ such that a^2-4 is a nonzero quadratic nonresidue. Now Corollary 1.2 is immediate from Theorem 1.1.

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