primes p	primitive closed geodesics $\mathcal{C}_0$
$p \mapsto \log(p)$	$\mathcal{C}_0 \mapsto \mathrm{length}(\mathcal{C}_0)$
$\pi(x) := \sum_{p \leq x} 1$	$\pi_\Gamma(x) := \sum_{\mathrm{length}(\mathcal{C}_0) \leq \log(x)} 1$
prime powers $n = p^k$	closed geodesics $\mathcal{C} = \mathcal{C}_0^k$
$n \mapsto \log(n)$	$\mathcal{C} \mapsto \mathrm{length}(\mathcal{C})$
$n \overset{\wedge}{\longmapsto} \log(p)$	$\mathcal{C} \overset{\wedge}{\longmapsto} \mathrm{length}(\mathcal{C}_0)$
$\psi(x) := \sum_{n \leq x} \Lambda(n)$	$\psi_\Gamma(x) := \sum_{\mathrm{length}(\mathcal{C}) \leq \log(x)} \Lambda(\mathcal{C})$

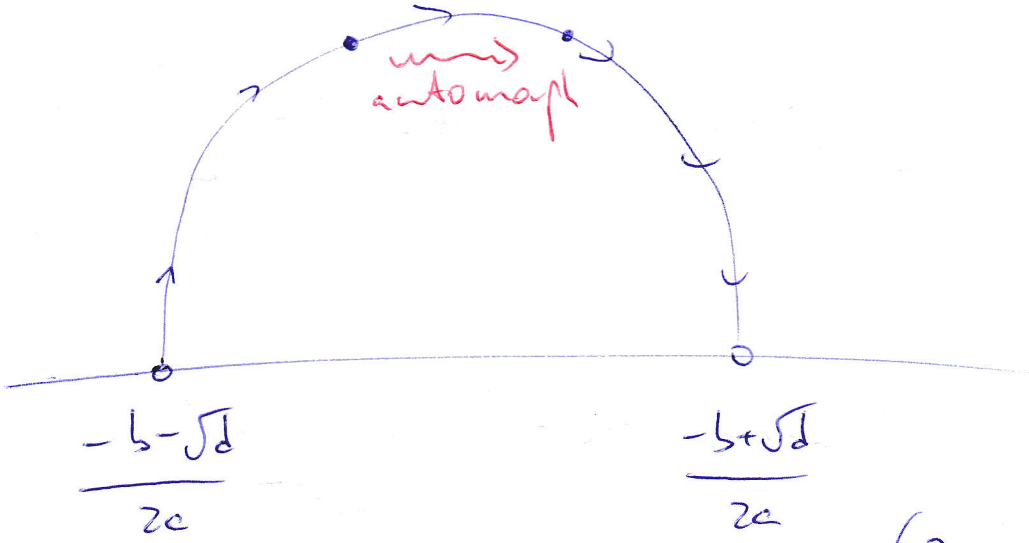
<u>Theorem</u> (Soundararajan – Young 2013)

$$\pi_\Gamma(x) = li(x) + O(x^{25/36+\varepsilon})$$

$$\psi_\Gamma(x) = x + O(x^{25/36+\varepsilon})$$

2

A closed geodesic $\mathcal{C}$ corresponds to an $SL_2(\mathbb{Z})$-orbit of a pair consisting of a primitive binary quadratic form $ax^2+bxy+cy^2$ with some positive (non-square) discriminant d, and an automorph $\begin{pmatrix} \frac{t-bu}{2} & -cu \\ au & \frac{t+bu}{2} \end{pmatrix} \in SL_2(\mathbb{Z})$ of the form, where $t, u > 0$ satisfy $t^2-du^2=4$:



If the automorph is conjugate to $\begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}$ with $\lambda > 1$ in $SL_2(\mathbb{R})$, then $\text{length}(\mathcal{C}) = \lg(\lambda^2)$, so the condition $\text{length}(\mathcal{C}) \leq \lg(x) \iff \lambda \leq \sqrt{x} \iff \lambda + \lambda^{-1} \leq \sqrt{x} + \frac{1}{\sqrt{x}}$

Therefore,

class number formula

$$\psi_\Gamma(x) = \sum_{3 \leq t \leq \sqrt{x}+\frac{1}{\sqrt{x}}} \; \sum_{\substack{du^2 = t^2-4 \\ d \text{ a positive discriminant} \\ u \geq 1}} 2\sqrt{d}\, L(1, \chi_d)$$

Now, for a given $t \geq 3$, let us write t^2-4 as $D\ell^2$, where D is a fund. discriminant. Then,

$$\sum_{\substack{du^2 = t^2-4 \\ d \text{ a discr.} \\ u \geq 1}} \sqrt{d}\, L(1, \chi_d) = \sum_{u \mid \ell} \sqrt{\frac{D\ell^2}{u^2}}\, L(1, \chi_{\frac{D\ell^2}{u^2}})$$

$$= \sqrt{D\ell^2} \sum_{u \mid \ell} L(1, \chi_{\frac{D\ell^2}{u^2}})\, u^{-1} = \sqrt{D\ell^2}\, L(1, D\ell^2),$$

where $L(s, D\ell^2) := \sum_{u \mid \ell} L(s, \chi_{\frac{D\ell^2}{u^2}})\, u^{1-2s}$.

$$= L(s, \chi_D) \sum_{u \mid \ell} u^{1-2s} \prod_{p \mid \frac{\ell}{u}} (1 - \chi_D(p) p^{-s}).$$

Now we determine the Euler factor at a prime p. If $p^r \| \ell$, then this factor equals

$$\sum_{0 \leq m < r} p^{m(1-2s)} + p^{r(1-2s)} (1 - \chi_D(p) p^{-s})^{-1},$$

whence

$$L(s, De^2) = \sum_{q=1}^{\infty} \lambda_q(De^2) q^{-s},$$

$q \mapsto \lambda_q(De^2)$ multiplicative,

$$\lambda_{p^k}(De^2) = \begin{cases} p^{k/2}, & 0 \le k \le 2r-2 \text{ even} \\ 0, & 1 \le k \le 2r-1 \text{ odd} \\ p^r \chi_D(p^{k-2r}), & k \ge 2r \end{cases}$$

To summarize so far, the above Dirichlet series $L(s, \delta)$ satisfies

<u>Theorem</u> (Bykovskii 1994, Kuznetsov 1978)

$$\Psi_\Gamma(x) = 2 \sum_{3 \le t \le \sqrt{x} + \frac{1}{\sqrt{x}}} \sqrt{t^2-4}\; L(1, t^2-4)$$

From the definition of $L(s, \delta)$, we also see that

$$\frac{L(s, De^2)}{L(s, \chi_D)} = \sum_{u \mid e} u^{1-2s} \sum_{f \mid \frac{e}{u}} \mu(f) \chi_D(f) f^{-s}$$

$$= \sum_{f \mid e} \mu(f) \chi_D(f) f^{-s} \sum_{u \mid \frac{e}{f}} u^{1-2s} .$$

Let us see what happens to RHS under $s \to 1-s$:

$$\sum_{f \mid \ell} \mu(f)\chi_D(f) f^{s-1} \sum_{u \mid \frac{\ell}{f}} u^{2s-1}$$

$$= \sum_{f \mid \ell} \mu(f)\chi_D(f) f^{s-1} \sum_{v \mid \frac{\ell}{f}} \left(\frac{\ell}{fv}\right)^{2s-1}$$

$$= \ell^{2s-1} \sum_{f \mid \ell} \mu(f)\chi_D(f) f^{-s} \sum_{v \mid \frac{\ell}{f}} v^{1-2s} \quad , \text{ whence}$$

$$\ell^{2s-1} \frac{L(s, D\ell^2)}{L(s, \chi_D)} = \frac{L(1-s, D\ell^2)}{L(1-s, \chi_D)}$$

$$\frac{\pi^{-\frac{s}{2}} (D\ell^2)^{\frac{s}{2}} \Gamma(\frac{s}{2}) L(s, D\ell^2)}{\pi^{-\frac{s}{2}} D^{\frac{s}{2}} \Gamma(\frac{s}{2}) L(s, \chi_D)} = \frac{\pi^{-\frac{1-s}{2}} (D\ell^2)^{\frac{1-s}{2}} \Gamma(\frac{1-s}{2}) L(1-s, D\ell^2)}{\pi^{-\frac{1-s}{2}} D^{\frac{1-s}{2}} \Gamma(\frac{1-s}{2}) L(1-s, \chi_D)}$$

Denominators are the same, therefore

$$\Lambda(s, \delta) := \pi^{-\frac{s}{2}} \delta^{\frac{s}{2}} \Gamma(\tfrac{s}{2}) L(s, \delta) \quad \text{satisfies}$$

$$\Lambda(s, \delta) = \Lambda(1-s, \delta)$$

In addition, we can see that

$$\text{GRH holds for } L(s, \chi_D) \iff$$
$$\text{GRH holds for } L(s, \delta).$$

Indeed,

$$\frac{L(s,\delta)}{L(s,\chi_D)} = \sum_{u \mid \ell} u^{1-2s} \prod_{p \mid \frac{\ell}{u}} (1 - \chi_D(p) p^{-s})$$

$$= \prod_{p^r \| \ell} \left\{ \sum_{0 \le m < r} p^{m(1-2s)} (1 - \chi_D(p) p^{-s}) + p^{r(1-2s)} \right\}$$

$$= \prod_{p^r \| \ell} \left\{ \sum_{0 \le m \le r} p^{m(1-2s)} - \chi_D(p) p^{-s} \sum_{0 \le m \le r-1} p^{m(1-2s)} \right\}$$

The Euler factor on RHS equals, upon setting $z = p^{\frac{1}{2}-s}$,

$$\sum_{0 \le m \le r} z^{2m} - \frac{\chi_D(p)}{\sqrt{p}} z \sum_{0 \le m \le r-1} z^{2m}.$$

It suffices to show that this polynomial has all its roots on the unit circle. For this, it suffices that it has a zero in each open arc

$$\left\{ z = e^{i\theta} : \pi \frac{j}{r} < \theta < \pi \frac{j+1}{r} \right\}, \quad j = 0, \dots, 2r-1$$

To see this, sum the polynomial via geometric series as (for $z^2 \neq 1$)

$$\frac{z^{2r+2}-1}{z^2-1} - \varepsilon z \frac{z^{2r}-1}{z^2-1}, \qquad \varepsilon := \frac{\chi_D(p)}{\sqrt{p}}.$$

This equals

$$z^r \frac{(z^{r+1}-z^{-r-1}) - \varepsilon(z^r - z^{-r})}{z - z^{-1}}.$$

For $z = e^{i\theta}$, numerator equals

$$2i(\sin(r+1)\theta - \varepsilon \sin r\theta)$$

$$= (2i \sin r\theta)\left(\frac{\sin(r+1)\theta}{\sin r\theta} - \varepsilon\right),$$

and second factor maps each of the arcs above bijectively onto $\mathbb{R}$.

We can also approximate $\psi_\Gamma(x)$ with the help of the Selberg trace formula

<u>Theorem</u> (Iwaniec 1984) For $1 \leq T \leq \frac{\sqrt{x}}{(\log x)^2}$ we have

$$\psi_\Gamma(x) = x + \sum_{|t_j| \leq T} \frac{x^{\frac{1}{2}+it_j}}{\frac{1}{2}+it_j} + O\left(\frac{x}{T}\log^2 x\right)$$

Soundararajan – Young then utilize two auxiliary results:

Theorem (Luo – Sarnak 1995) For $r \geq 2$ we have

(1) $$\sum_{|t_j| \leq T} r^{it_j} \ll T^{\frac{5}{4}} r^{\frac{1}{8}} \log^2 T$$

Theorem (Conrey – Iwaniec 2000)

(2) $$L(\tfrac{1}{2}+it, \chi_D) \ll_\varepsilon (1+|t|)^A |D|^{\theta+\varepsilon}$$

for some $A > 0$ and $\theta := \frac{1}{6}$

They choose a nice bump function

for some $x^{\frac{1}{2}+\varepsilon} \leq Y \leq x^{1-\varepsilon}$ and find that

$$\Psi_\Gamma(x) = \int_0^Y \Psi_\Gamma(x+u)h(u)du - \int_0^Y (\Psi_\Gamma(x+u) - \Psi_\Gamma(x))h(u)du$$

$$= \left\{ x + \int_0^Y u\,h(u)\,du + O(x^{\frac{7}{8}+\varepsilon} Y^{-\frac{1}{4}}) \right\}$$ ← Thm. 3.1, based on (1)

$$- \int_0^Y \left\{ u + O(u^{\frac{1}{2}} x^{\frac{1}{4}+\frac{\theta}{2}}) \right\} h(u)$$ ← Thm. 3.2, based on (2)

$$= x + O(x^{\frac{7}{8}+\varepsilon} Y^{-\frac{1}{4}}) + O(x^{\frac{1}{4}+\frac{\theta}{2}+\varepsilon} Y^{\frac{1}{2}})$$

Optimal choice is $Y^{\frac{3}{4}} = x^{\frac{5}{8}-\frac{\theta}{2}}$, i.e., $Y = x^{\frac{5}{6}-\frac{2\theta}{3}}$.

This gives

$$\psi_\Gamma(x) = x + O(x^{\frac{1}{4}+\frac{\theta}{2}+\varepsilon} x^{\frac{5}{12}-\frac{\theta}{3}})$$

$$= x + O(x^{\frac{2}{3}+\frac{\theta}{6}+\varepsilon}).$$

For $\theta = \frac{1}{6}$ this is the theorem, while $\theta = 0$ (Lindelöf) yields $x + O(x^{\frac{2}{3}+\varepsilon})$.

<u>Proof of Thm. 3.1:</u> Using the result of Iwaniec, based on the Selberg trace formula,

$$\int_0^Y \psi_\Gamma(x+u)\, k(u)\, du = x + \int_0^Y u\, k(u)\, du$$

$$+ \sum_{|t_j| \leq T} \frac{1}{\frac{1}{2}+it_j} \int_0^Y (x+u)^{\frac{1}{2}+it_j} k(u)\, du + O\left(\frac{x}{T} \log^2 x\right)$$

for any $1 \leq T \leq \frac{\sqrt{x}}{(\log x)^2}$. Choosing $T := \frac{\sqrt{x}}{(\log x)^3}$, it suffices to show that

$$\sum_{|t_j| \leq \frac{\sqrt{x}}{(\log x)^3}} \frac{1}{\frac{1}{2}+it_j} \int_0^y (x+u)^{\frac{1}{2}+it_j} h(u)\, du \ll x^{\frac{7}{8}+\varepsilon} y^{-\frac{1}{4}}$$

Integrating by parts ℓ times,

$$\int_0^y (x+u)^{\frac{1}{2}+it_j} h(u)\, du = (-1)^\ell \int_0^y \frac{(x+u)^{\ell+\frac{1}{2}+it_j}}{(\frac{3}{2}+it_j)\cdots(\ell+\frac{1}{2}+it_j)} h^{(\ell)}(u)\, du$$

$$\ll \frac{x^{\frac{1}{2}+\ell}}{(|t_j| y)^\ell} . \quad \text{So}$$

$$\sum_{|t_j|>Z} \frac{1}{\frac{1}{2}+it_j} \int_0^y \ldots \ll \sum_{|t_j|>Z} \frac{x^{\frac{1}{2}+\ell}}{|t_j|^{\ell+1} y^\ell}$$

$$\ll Z x^{\frac{1}{2}} \left(\frac{x}{Zy}\right)^\ell, \text{ so } Zy = x^{1+\varepsilon}, \text{ i.e.}$$

$Z := x^{1+\varepsilon}/y$ contributes a negligible amount. The remaining contribution is, using $S(t) := \sum_{|t_j| \leq t} v^{it_j}$, $v := x+u$,

$$\sum_{|t_j| \leq Z} \frac{v^{\frac{1}{2}+it_j}}{\frac{1}{2}+it_j}$$

$$= v^{\frac{1}{2}} \int_1^Z \frac{dS(t)}{\frac{1}{2}+it} = v^{\frac{1}{2}} \left[\frac{S(t)}{\frac{1}{2}+it}\right]_1^Z + v^{\frac{1}{2}} i \int_1^Z \frac{S(t)}{(\frac{1}{2}+it)^2} dt \ll x^{\frac{5}{8}} Z^{\frac{1}{4}+\varepsilon}.$$

Let us convolve $q \mapsto \lambda_q(De^2)$ with $q \mapsto \mu^2(q)$.

The value at p^k equals

$$\begin{cases} p^{k/2}, & 0 \leq k \leq 2r \text{ even} \\ p^{(k-1)/2}, & 1 \leq k \leq 2r-1 \text{ odd} \\ p^r \chi_D(p^{k-2r}) + p^r \chi_D(p^{k-1-2r}), & k \geq 2r+1. \end{cases}$$

This is the same as

$$\begin{cases} p^{\lfloor k/2 \rfloor}, & 0 \leq k \leq 2r \\ p^r, & k = 2r+1, \quad p \mid D \\ 0, & k \geq 2r+2, \quad p \mid D \\ 0, & k \geq 2r+1, \quad \chi_D(p) = -1 \\ 2p^r, & k \geq 2r+1, \quad \chi_D(p) = 1 \end{cases}$$

We claim that this is the same as

$$\varrho_{p^k}(De^2) := \#\{x \pmod{2p^k} : x^2 \equiv De^2 \pmod{4p^k}\}$$

Indeed, assume first that $k \leq 2r$ and p is odd. Then by $p^{\nu} \| \ell$ we get

$$\varrho_{p^k}(D\ell^2) = \#\{x \pmod{p^k} : x^2 \equiv 0 \pmod{p^k}\}$$

$$= p^{k-\lceil k/2 \rceil} = p^{\lfloor k/2 \rfloor}.$$

For $k \leq 2\nu - 2$ and $p = 2$ the same argument works with $2^{k+1-\lceil (k+2)/2 \rceil} = 2^{\lfloor k/2 \rfloor}$ on RHS. For $k = 2\nu - 1$ and $p = 2$ we get

$$\varrho_{p^k}(D\ell^2) = \#\{x \pmod{2^{2\nu}} : x^2 \equiv D\ell^2 \pmod{2^{2\nu+1}}\}$$

$$= \#\{y \pmod{2^{\nu}} : y^2 \equiv D \pmod{2}\}$$

$$= 2^{\nu-1} = 2^{\lfloor k/2 \rfloor}.$$

For $k = 2\nu$ and $p = 2$ we get

$$\varrho_{p^k}(D\ell^2) = \#\{x \pmod{2^{2\nu+1}} : x^2 \equiv D\ell^2 \pmod{2^{2\nu+2}}\}$$

$$= \#\{y \pmod{2^{\nu+1}} : y^2 \equiv D \pmod{4}\}$$

$$= 2^{\nu} = 2^{\lfloor k/2 \rfloor}.$$

So we are done with the $k \leq 2\nu$ case.

Now let $k \geq 2\nu + 1$ ~~$p \nmid D$~~. Then clearly

$$\varrho_{p^k}(D\ell^2) = \#\{y \pmod{2p^{k-\nu}} : y^2 \equiv D \pmod{4p^{k-2\nu}}\}$$

Let $k=2r+1$ and $p \mid D$. Then the count is

$$\#\{y \pmod{2p^{r+1}} : y^2 \equiv D \pmod{4p}\}.$$

If p is odd, then this is just

$$\#\{y \pmod{p^{r+1}} : y \equiv 0 \pmod{p}\} = p^r.$$

If $p=2$, then the count is

$$\#\{y \pmod{2^{r+2}} : y^2 \equiv D \pmod{8}\}$$

$$= \#\{z \pmod{2^{r+1}} : z^2 \equiv \frac{D}{4} \pmod{2}\} = 2^r.$$

Let $k \geq 2r+2$ and $p \mid D$. If p is odd, then $y^2 \equiv D \pmod{p^2}$ is a contradiction, because $p \parallel D$.

If $p=2$, then $y^2 \equiv D \pmod{2^4}$ yields $\left(\frac{y}{2}\right)^2 \equiv \frac{D}{4} \pmod{4}$, which is a contradiction, because $\frac{D}{4} \equiv 2, 3 \pmod{4}$. Hence

$$\varrho_{p^k}(De^2) = 0 \quad \text{for } k \geq 2r+2 \text{ and } p \mid D,$$

and also

$$\varrho_{p^k}(De^2) = p^r \quad \text{for } k = 2r+1 \text{ and } p \mid D$$

Now let $k \geq 2r+1$ and $p \nmid D$.

If p is odd, then the count is

$$\rho_{p^k}(D\ell^2) = \#\{y \ (\mathrm{mod}\ p^{k-r}) : y^2 \equiv D \ (\mathrm{mod}\ p^{k-2r})\}$$

For $\chi_D(p) = -1$, the congruence has no solution.

For $\chi_D(p) = 1$, the congruence has 2 solutions mod p^{k-2r}, hence $2p^r$ solutions mod p^{k-r}.

If $p = 2$, then the count is

$$\rho_{2^k}(D\ell^2) = \#\{y \ (\mathrm{mod}\ 2^{k-r+1}) : y^2 \equiv D \ (\mathrm{mod}\ 2^{k-2r+2})\}$$

For $\chi_D(2) = -1$ we have $D \equiv \pm 3 \ (\mathrm{mod}\ 8)$, so the congruence has no solution.

For $\chi_D(2) = 1$ we have $D \equiv \pm 1 \ (\mathrm{mod}\ 8)$, hence in fact $D \equiv 1 \ (\mathrm{mod}\ 8)$. In this case, the congruence has 4 solutions mod 2^{k-2r+2}, hence $4 \cdot 2^{r-1} = 2^{r+1}$ solutions mod 2^{k-r+1}.

To summarize,

$$\rho_{p^h}(\Delta\ell^2) = \begin{cases} 0, & h \geq 2\nu+1, \quad \chi_\Delta(p) = -1 \\ 2p^\nu, & h \geq 2\nu+1, \quad \chi_\Delta(p) = 1. \end{cases}$$

This proves that

$$L(s, \Delta\ell^2) = \sum_{q=1}^{\infty} \lambda_q(\Delta\ell^2)\, q^{-s}$$

$$= \frac{\zeta(2s)}{\zeta(s)} \sum_{q=1}^{\infty} \rho_q(\Delta\ell^2)\, q^{-s} .$$

Equivalently,

$$(3) \qquad \lambda_q(\Delta\ell^2) = \sum_{q_1^2 q_2 q_3 = q} \mu(q_2)\, \rho_{q_3}(\Delta\ell^2) .$$

Here, $\rho_{q_3}(\Delta\ell^2)$ counts the solutions $y \pmod{2q_3}$ of the congruence $y^2 \equiv \Delta\ell^2 \pmod{4q_3}$. If $\Delta\ell^2 = t^2 - 4$ is the discriminant occurring in Kuznetsov's formula, we have $y \equiv t \pmod{2}$, so we can write $y = 2y_1 + t$, where $y_1 \pmod{\rho_3}$ is determined by $y \pmod{2\rho_3}$. Then,

$$(2y_1+t)^2 \equiv t^2 - 4 \pmod{4\rho_3},$$

i.e.,

$$y_1^2 + y_1 t + 1 \equiv 0 \pmod{\rho_3}.$$

In other words, y_1 is coprime to ρ_3, and $t \equiv -y_1 - \bar{y}_1 \pmod{\rho_3}$. That is,

$$S_{\rho_3}(t^2-4) = \sum_{y_1 \pmod{\rho_3}} \frac{1}{\rho_3} \sum_{k \pmod{\rho_3}} e\left(k\frac{t+y_1+\bar{y}_1}{\rho_3}\right)$$

$$= \frac{1}{\rho_3} \sum_{k \pmod{\rho_3}} e\left(\frac{kt}{\rho_3}\right) S(k,k;\rho_3).$$

Using Selberg's identity for Kloosterman sums (stated by Selberg in 1938, proved by Kuznetsov in 1981 using his famous formula, and proved by Matthes in 1990 using elementary manipulations) we get

$$\rho_{\rho_3}(t^2-4) = \frac{1}{\rho_3}\sum_{k \,(\mathrm{mod}\, \rho_3)} e\left(\frac{kt}{\rho_3}\right)\sum_{d \mid (k,\rho_3)} d\, S\left(\frac{k^2}{d^2}, 1; \frac{\rho_3}{d}\right)$$

Writing $k = dm$, $\rho_3 = d\rho_4$, this becomes

$$\rho_{\rho_3}(t^2-4) = \sum_{\rho_4 \mid \rho_3} \frac{1}{\rho_4} \sum_{m \,(\mathrm{mod}\, \rho_4)} e\left(\frac{mt}{\rho_4}\right) S(m^2, 1; \rho_4)$$

Combining with (3), we obtain

$$\lambda_\rho(t^2-4) = \sum_{\rho_1^2\rho_2\rho_4 \mid \rho} \frac{\mu(\rho_2)}{\rho_4} \sum_{m \,(\mathrm{mod}\, \rho_4)} e\left(\frac{mt}{\rho_4}\right) S(m^2, 1; \rho_4)$$

$$= \sum_{\rho_1^2\rho_4 \mid \rho} \frac{1}{\rho_4} \sum_{m \,(\mathrm{mod}\, \rho_4)} e\left(\frac{mt}{\rho_4}\right) S(m^2, 1; \rho_4)$$

$$\times \sum_{\rho_2 \mid \frac{\rho}{\rho_1^2\rho_4}} \mu(\rho_2)$$

$$= \sum_{\rho_1^2\rho_4 = \rho} \frac{1}{\rho_4} \sum_{m \,(\mathrm{mod}\, \rho_4)} e\left(\frac{mt}{\rho_4}\right) S(m^2, 1; \rho_4).$$

<u>Lemma</u> If we write q as $a^2 b$ with b square-free, for any $z \geq 2$ we have

$$\sum_{t \leq z} \lambda_q(t^2-4) = z \frac{\mu(b)}{b} + O(q^{\frac{1}{2}+\varepsilon}).$$

<u>Proof</u> By the previous identity,

$$\sum_{t \leq z} \lambda_q(t^2-4) = \sum_{q_7^2 q_4 = q} \frac{1}{q_4} \sum_{m \,(\mathrm{mod}\, q_4)} S(m^2, 1; q_4)$$

$$\times \sum_{t \leq z} e\left(\frac{mt}{q_4}\right).$$

For $m=0$ the inner sum is $z + O(1)$, while $S(m^2, 1; q_4) = \mu(q_4)$, which is nonzero if and only if $q_4 = b$. For $m \neq 0$ the inner sum is $\ll \left\|\frac{m}{q_4}\right\|^{-1}$, while $S(m^2, 1; q_4)$ is $\ll q_4^{1/2+\varepsilon}$. Hence we get

$$\sum_{t \leq z} \lambda_q(t^2-4) = (z + O(1)) \frac{\mu(b)}{b} + O\Bigg(\sum_{q_4 | q} q_4^{1/2+\varepsilon} \sum_{1 \leq m \leq q_4 - 1} \frac{1}{q_4 \left\|\frac{m}{q_4}\right\|}$$

$$= z\frac{\mu(b)}{b} + O(q^{1/2+\varepsilon}).$$

Proof of Thm. 3.2: Using Kuznetsov's formula for the prime geodesic counting function,

$$\psi_\Gamma(x+u)-\psi_\Gamma(x) = 2\sum_{X\leq t\leq X'} \sqrt{t^2-4}\, L(1,t^2-4),$$

where $X := \sqrt{x}+\frac{1}{\sqrt{x}}$, $X' := \sqrt{x+u}+\frac{1}{\sqrt{x+u}}$.

Here, $\sqrt{t^2-4} = t\left(1+O\left(\frac{1}{t^2}\right)\right) = t\left(1+O\left(\frac{1}{x}\right)\right)$, whence

$$\psi_\Gamma(x+u)-\psi_\Gamma(x) = \left(2+O\left(\frac{1}{x}\right)\right)\sum_{X<t\leq X'} t\, L(1,t^2-4).$$

We express $L(1,t^2-4)$ via (where $V\geq 1$ is arbitrary)

$$S_V(t^2-4) := \sum_{q=1}^{\infty}\frac{\lambda_q(t^2-4)}{q}e^{-q/V}$$

$$= \frac{1}{2\pi i}\int_{(1)} L(1+s,t^2-4)\,V^s\,\Gamma(s) = L(1,t^2-4) + \frac{1}{2\pi i}\int_{(-\frac{1}{2})}\dots$$

Using the Conrey-Iwaniec bound (2), the last integral is $O(V^{-\frac{1}{2}} t^{2\vartheta+\varepsilon})$, so that

$$L(1, t^2-4) = S_V(t^2-4) + O(V^{-\frac{1}{2}} t^{2\vartheta+\varepsilon})$$

This yields

$$\psi_\Gamma(x+u) - \psi_\Gamma(x) = \left(2 + O\left(\tfrac{1}{x}\right)\right) \sum_{x < t \leq x'} t\, S_V(t^2-4)$$

$$+ O(u V^{-\frac{1}{2}} x^{2\vartheta+\varepsilon})$$

The sum here equals

$$\sum_{x < t \leq x'} t \sum_{q=1}^{\infty} \frac{\lambda_q(t^2-4)}{q} e^{-q/V}$$

$$= \sum_{q=1}^{\infty} \frac{e^{-q/V}}{q} \left\{ \sum_{x < t \leq x'} t\, \lambda_q(t^2-4) \right\}.$$

Inner sum equals, by the Lemma and partial summation,

$$\{\ldots\} = \int_X^{X'} z \, d\Big(\sum_{t \le z} \lambda_q(t^2 - q) \Big)$$

$$= \Big[z \sum_{t \le z} \lambda_q(t^2-q) \Big]_X^{X'} - \int_X^{X'} \sum_{t \le z} \lambda_q(t^2-q) \, dt$$

$$= \Big[z \Big(z \frac{\mu(b)}{b} + O\big(q^{1/2+\varepsilon}\big) \Big) \Big]_X^{X'} - \int_X^{X'} \Big(z \frac{\mu(b)}{b} + O\big(q^{1/2+\varepsilon}\big) \Big) dt$$

$$= \frac{X'^2 - X^2}{2} \frac{\mu(b)}{b} + O\big(X q^{1/2+\varepsilon}\big)$$

$$= \frac{u}{2} \frac{\mu(b)}{b} + O\big(X q^{1/2+\varepsilon}\big)$$

Therefore,

$$\psi_r(x+u) - \psi_r(x) = \Big(1 + O\big(\tfrac{1}{x}\big)\Big) u \sum_{a,b} \mu(b) \frac{e^{-a^2 b/v}}{a^2 b^2} + O\big(X v^{\frac{1}{2}+\varepsilon}\big)$$
$$+ O\big(u v^{-\frac{1}{2}} X^{2\theta+\varepsilon}\big)$$

The last sum can be expressed as

$$\sum_{a,b}\mu(b)\frac{e^{-a^2b/U}}{a^2b^2} = \sum_{a,b}\frac{\mu(b)}{a^2b^2}\frac{1}{2\pi i}\int_{(1)}\left(\frac{a^2b}{U}\right)^{-s}\Gamma(s)\,ds$$

$$= \frac{1}{2\pi i}\int_{(1)}\left(\sum_{a,b}\frac{\mu(b)}{a^{2+2s}b^{2+s}}\right)U^s\Gamma(s)\,ds$$

$$= \frac{1}{2\pi i}\int_{(1)}\frac{\zeta(2+2s)}{\zeta(2+s)}U^s\Gamma(s)\,ds$$

$$\overset{\text{pólus } s=0 \text{ és } s=-\frac{1}{2} \text{ esetén}}{=} 1 + \text{const.}\,U^{-\frac{1}{2}} + O\left(U^{-\frac{2}{3}}\right)$$

$$= 1 + O\left(U^{-\frac{1}{2}}\right), \text{ so}$$

$$\psi_r(x+u)-\psi_r(x) = \left(1+O\left(\tfrac{1}{x}\right)\right)u\left(1+O\left(U^{-\frac{1}{2}}\right)\right) + O\left(uU^{-\frac{1}{2}}x^{2\theta+\varepsilon}\right) + O\left(xU^{\frac{1}{2}+\varepsilon}\right)$$

$$= u + O\left(xU^{\frac{1}{2}+\varepsilon} + uU^{-\frac{1}{2}}x^{2\theta+\varepsilon}\right)$$

Now we choose $U := u\,x^{-1+2\theta}$, and we get

$$= u + O\left(u^{\frac{1}{2}}\underbrace{x^{\frac{1}{2}+\theta}}_{x^{\frac{1}{2}+\frac{\theta}{2}}}\right).$$