Beyond the spherical sup-norm problem I–II.

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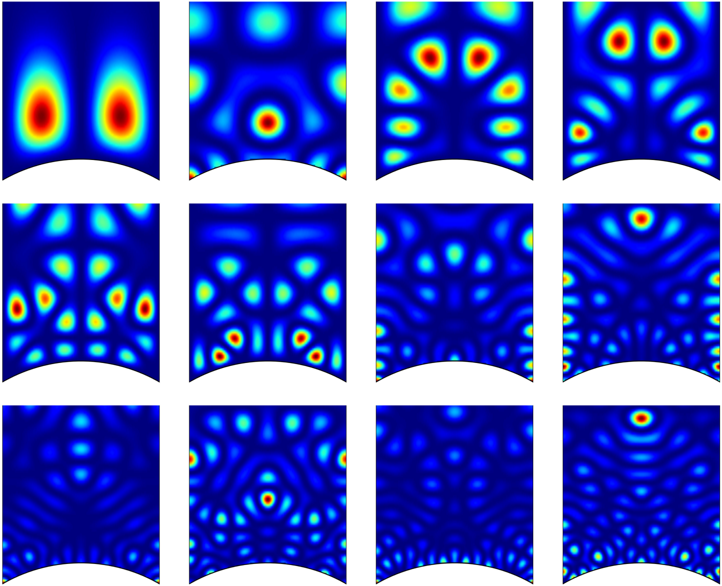
Maass forms on the modular surface



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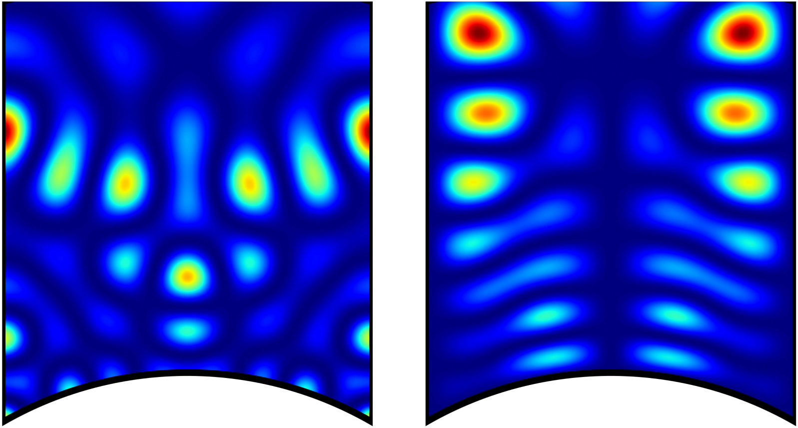
Maass forms with $\lambda \approx 10^3$ on the modular surface



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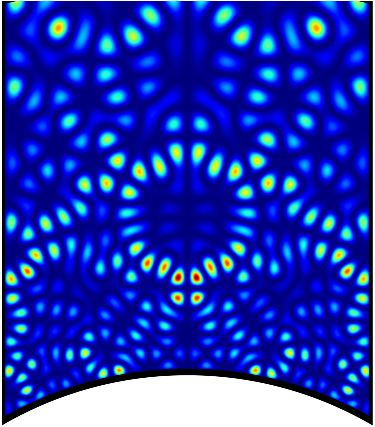
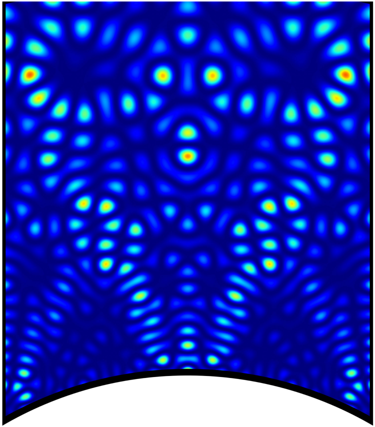
Maass forms with $\lambda \approx 10^4$ on the modular surface





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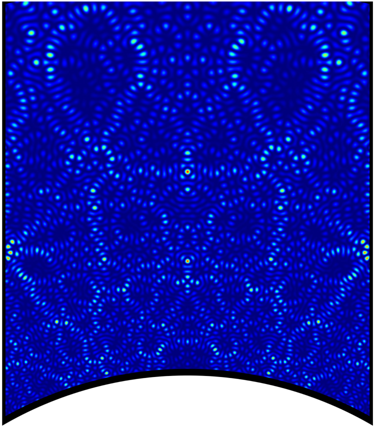
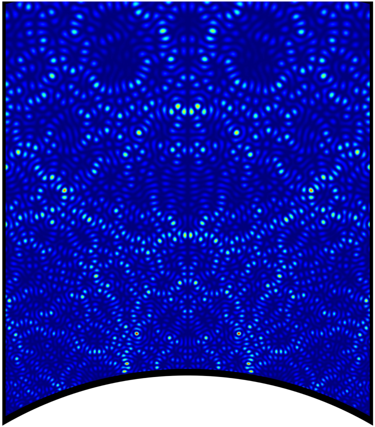
Maass forms with $\lambda \approx 10^5$ on the modular surface





Fredrik Strömberg

The upper half-plane as a quotient of $\mathrm{SL}_2(\mathbb{R})$

Notation

$$G := \mathrm{SL}_2(\mathbb{R}), \qquad K := \mathrm{SO}_2(\mathbb{R}), \qquad \Gamma := \mathrm{SL}_2(\mathbb{Z}),$$
$$\mathcal{H} := \{x + yi : \ x \in \mathbb{R}, \ y > 0\}.$$

The Lie group G acts on the upper half-plane $\mathcal{H}$ transitively via

$$gP := \underbrace{(aP+b)(cP+d)^{-1}}_{\text{arithmetic in complex numbers}}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, \quad P \in \mathcal{H}.$$

The stabilizer of $i \in \mathcal{H}$ is the maximal compact subgroup $K \leqslant G$, hence we can identify each point $gi \in \mathcal{H}$ with the coset $gK \subset G$. So we can identify $\mathcal{H} \cong G/K$, hence also $\Gamma\backslash\mathcal{H} \cong \Gamma\backslash G/K$.

The (two-sided) Haar measure on G gives rise to a G-invariant measure on $\mathcal{H}$, and we can identify $L^2(\Gamma\backslash\mathcal{H})$ with the set of those elements of $L^2(\Gamma\backslash G)$ that are fixed by the right action of K. This closed subspace of $L^2(\Gamma\backslash G)$ is the space of spherical vectors.

Spectral decomposition of the automorphic L^2 space

The advantage of $L^2(\Gamma\backslash G)$ is that G acts on it (from the right) by unitary operators. By abstract theory (using only properties of G), this space decomposes uniquely as a direct integral of irreducible unitary representations of G:

$$L^2(\Gamma\backslash G) = \int_{\widehat{G}} V_\pi \,\mathrm{d}\mu_{\mathrm{sp}}(\pi).$$

This goes back to the work of Murray–Neumann (1936) and Mautner (1950). Following the more concrete approach of Roelcke (1955) and Selberg (1956), we also have an orthodecomposition

$$L^2(\Gamma\backslash G) = \mathbb{C} \oplus L^2_{\mathrm{cusp}}(\Gamma\backslash G) \oplus L^2_{\mathrm{Eis}}(\Gamma\backslash G).$$

The Eisenstein subspace is rather explicit, and the restriction of $\mathrm{d}\mu_{\mathrm{sp}}(\pi)$ to it has no point masses. In contrast, the cuspidal subspace is elusive, and the restriction of $\mathrm{d}\mu_{\mathrm{sp}}(\pi)$ to it is a sum of finite point masses. In other words, $L^2_{\mathrm{cusp}}(\Gamma\backslash G)$ is an orthogonal direct sum of irreducible G-spaces V_π, each with finite multiplicity, and the elements of these constituents are the cusp forms for $\Gamma\backslash G$.

The unitary dual of $\mathrm{SL}_2(\mathbb{R})$

The unitary dual $\widehat{G}$ was determined by Bargman (1947), perhaps inspired by the work of Wigner (1939). The nontrivial irreducible unitary representations of G are infinite dimensional, and the ones relevant here (the tempered ones) come in 4 families:

- spherical/non-spherical principal series $\pi_{it}^{\pm}$ for $t \in \mathbb{R}_{\geqslant 0}$;
- holomorphic/antiholomorphic discrete series $\pi_k^{\pm}$ for $k \in \mathbb{Z}_{\geqslant 1}$.

These representations can be defined explicitly, e.g. by letting G act on $L^2(\mathbb{R})$ in natural but different ways. We can clearly distinguish between the above 4 types by looking at how they decompose into irreducible K-spaces (we parametrize $\widehat{K}$ by $\mathbb{Z}$):

$$V_{it}^{+} = \bigoplus_{\substack{\ell \in \mathbb{Z} \\ \ell \equiv 0 \bmod 2}} V_{it}^{+,\ell} \qquad V_{it}^{-} = \bigoplus_{\substack{\ell \in \mathbb{Z} \\ \ell \equiv 1 \bmod 2}} V_{it}^{-,\ell}$$

$$V_{k}^{+} = \bigoplus_{\substack{\ell \geqslant k \\ \ell \equiv k \bmod 2}} V_{k}^{+,\ell} \qquad V_{k}^{-} = \bigoplus_{\substack{\ell \leqslant -k \\ \ell \equiv k \bmod 2}} V_{k}^{-,\ell}$$

The summands here are one-dimensional (i.e. isomorphic to $\mathbb{C}$).

The sup-norm problem for $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ (1 of 2)

It is now clear that π_{it}^{+} has a spherical vector (unique up to scaling), while π_{it}^{-} and $\pi_{k}^{\pm}$ have not. If π_{it}^{+} occurs in $L^2_{\mathrm{cusp}}(\Gamma\backslash G)$, then its spherical vector is a classical Maass form of weight zero and Laplacian eigenvalue $\frac{1}{4}+t^2$. The spherical sup-norm problem aims at bounding the sup-norm of these Maass forms non-trivially.

Theorem (Iwaniec–Sarnak 1995)

Let $G=\mathrm{SL}_2(\mathbb{R})$ and $\Gamma=\mathrm{SL}_2(\mathbb{Z})$. Let ϕ be an L^2-normalized spherical Hecke–Maass form on $\Gamma\backslash G$ of Laplacian eigenvalue λ. Then for any $\varepsilon>0$ we have $\|\phi\|_\infty \ll_\varepsilon \lambda^{5/24+\varepsilon}$.

This result was generalized in a natural fashion from $\mathbb{Z}$ to $\mathbb{Z}[i]$ by Blomer–Harcos–Milićević (2016), and then to the ring of integers of any number field by Blomer–Harcos–Maga–Milićević (2020). I will return to this later. Strong results in the level aspect are also available, but will not be discussed in these lectures.

The sup-norm problem for $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ (2 of 2)

Inspired by the theory of newforms in the level aspect, the natural non-spherical sup-norm problem would concern a minimal weight vector in π_k^+ (or equivalently a maximal weight vector in π_k^-), which is again unique up to scaling. However, such a vector has weight k Laplacian eigenvalue $\frac{k}{2}\left(1-\frac{k}{2}\right)$, which grows with k. So the weight aspect is not separated from the eigenvalue aspect in this variant of the problem.

To go genuinely beyond the spherical sup-norm problem, we are led to work with $\mathrm{SL}_2(\mathbb{C})$ rather than $\mathrm{SL}_2(\mathbb{R})$.

If π_k^+ occurs in $L^2_{\mathrm{cusp}}(\Gamma\backslash G)$, then *the absolute value* of its minimal weight vector is invariant under K, and as a function on $\mathcal{H}$ it agrees with $F(x+iy) := y^{k/2}|f(x+iy)|$, where f is a holomorphic cusp form of weight k and level 1 on $\mathcal{H}$. Assuming that f is a Hecke eigenform, we have $\|F\|_\infty/\|F\|_2 \ll_\varepsilon k^{1/4+\varepsilon}$ by a result of Xia (2007). For co-compact $\Gamma \leqslant \mathrm{SL}_2(\mathbb{R})$, a similar result was proved by Khayutin–Steiner (2020), improving on Das–Sengupta (2015).

The upper half-space as a quotient of $\mathrm{SL}_2(\mathbb{C})$

Notation

$$G := \mathrm{SL}_2(\mathbb{C}), \qquad K := \mathrm{SU}_2(\mathbb{C}), \qquad \Gamma := \mathrm{SL}_2(\mathbb{Z}[i]),$$
$$\mathcal{H} := \{x + yi + zj : \ x, y \in \mathbb{R}, \ z > 0\}.$$

The Lie group G acts on the upper half-space $\mathcal{H}$ transitively via

$$gP := \underbrace{(aP+b)(cP+d)^{-1}}_{\text{arithmetic in quaternions}}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, \quad P \in \mathcal{H}.$$

The stabilizer of $j \in \mathcal{H}$ is the maximal compact subgroup $K \leqslant G$, hence we can identify each point $gj \in \mathcal{H}$ with the coset $gK \subset G$. So we can identify $\mathcal{H} \cong G/K$, hence also $\Gamma \backslash \mathcal{H} \cong \Gamma \backslash G/K$.

Now things are as before, $L^2(\Gamma\backslash\mathcal{H})$ is the space of spherical vectors in $L^2(\Gamma\backslash G)$, while $L^2(\Gamma\backslash G)$ decomposes uniquely as a direct integral of irreducible unitary representations of G. However, the unitary dual of $\mathrm{SL}_2(\mathbb{C})$ is quite different from that of $\mathrm{SL}_2(\mathbb{R})$.

The unitary dual of $\mathrm{SL}_2(\mathbb{C})$

The unitary dual $\widehat{G}$ was determined by Gelfand–Naimark (1947). The nontrivial irreducible unitary representations of G are infinite dimensional, and the ones relevant for the moment (the tempered ones) come in a single family:

- principal series $\pi_{it,p}$ for $t \in \mathbb{R}_{\geqslant 0}$ and $p \in \frac{1}{2}\mathbb{Z}$.

These representations can be defined explicitly, e.g. by letting G act on $L^2(\mathbb{C})$ in a natural way.

It is instructive (and crucial for us) to look at how these representations decompose into irreducible K-spaces (we parametrize $\widehat{K}$ by $\frac{1}{2}\mathbb{Z}_{\geqslant 0}$), and further into one-dimensional subspaces under the action of the diagonal subgroup of K:

$$V_{it,p} = \bigoplus_{\substack{\ell \geqslant |p| \\ \ell \equiv p \bmod 1}} V_{it,p}^{\ell} = \bigoplus_{\substack{\ell \geqslant |p| \\ \ell \equiv p \bmod 1}} \bigoplus_{\substack{|q| \leqslant \ell \\ q \equiv \ell \bmod 1}} V_{it,p}^{\ell,q}.$$

Here $\dim V_{it,p}^{\ell} = 2\ell + 1$ and $\dim V_{it,p}^{\ell,q} = 1$.

The sup-norm problem for $\mathrm{SL}_2(\mathbb{Z}[i])\backslash\mathrm{SL}_2(\mathbb{C})$

It is now clear that $\pi_{it,p}$ has a spherical vector (unique up to scaling) when $p=0$, and no spherical vector when $p\neq 0$. If $\pi_{it,0}$ occurs in $L^2_{\mathrm{cusp}}(\Gamma\backslash G)$, then its spherical vector is a Maass form of weight zero and Laplacian eigenvalue $1+t^2$. The spherical sup-norm problem aims at bounding the sup-norm of these Maass forms non-trivially.

Theorem (Blomer–Harcos–Milićević 2016)

Let $G=\mathrm{SL}_2(\mathbb{C})$ and $\Gamma=\mathrm{SL}_2(\mathbb{Z}[i])$. Let ϕ be an L^2-normalized spherical Hecke–Maass form on $\Gamma\backslash G$ of Laplacian eigenvalue λ. Then for any $\varepsilon>0$ we have $\|\phi\|_\infty \ll_\varepsilon \lambda^{5/12+\varepsilon}$.

This theorem is a Gauss integer analogue of the celebrated result of Iwaniec–Sarnak (1995). It was generalized further to the ring of integers of any number field by Blomer–Harcos–Maga–Milićević (2020). The cusp forms in this generalization are Γ-invariant eigenfunctions on a product of upper half-planes and half-spaces.

New results

Notation

$$G := \mathrm{SL}_2(\mathbb{C}), \qquad K := \mathrm{SU}_2(\mathbb{C}), \qquad \Gamma := \mathrm{SL}_2(\mathbb{Z}[i]).$$

Theorem (Blomer–Harcos–Maga–Milićević 2021)

Let $\ell \geqslant 1$ be an integer, $I \subset \mathbb{R}$ and $\Omega \subset G$ be compact sets. Let $V_\pi \subset L^2_{\mathrm{cusp}}(\Gamma \backslash G)$ be a cuspidal representation such that $\pi \simeq \pi_{it,p}$, where $t \in I$ and $|p| = \ell$. As usual, we assume that V_π consists of Hecke eigenfunctions. Let us choose an orthonormal basis $\{\phi_q : |q| \leqslant \ell\}$ of V_π^ℓ, with $\phi_q \in V_\pi^{\ell,q}$. Then for any $\varepsilon > 0$ we have

$$\sum_{|q| \leqslant \ell} |\phi_q(g)|^2 \ll_{\varepsilon,I,\Omega} \ell^{8/3+\varepsilon}, \qquad g \in \Omega.$$

For the individual summands we have

$$\phi_q(g) \ll_{\varepsilon,I,\Omega} \ell^{26/27+\varepsilon}, \qquad g \in \Omega.$$

Finally, for $q = 0$ (resp. for $q = \pm\ell$ under a technical assumption), we can improve the exponent to $7/8 + \varepsilon$ (resp. to $1/2 + \varepsilon$).

The *-algebra $L^1(G)$

Consider the convolution algebra $L^1(G)$ with the involution

$$f^*(g) := \overline{f(g^{-1})}, \qquad g \in G.$$

This is an anti-automorphism, making $L^1(G)$ into a *-algebra.

Each $\pi \in \widehat{G}$ determines a non-degenerate representation of $L^1(G)$ on V_π and vice versa:

$$\pi(f) := \int_G f(g)\pi(g)\,\mathrm{d}g \in \mathrm{End}(V_\pi).$$

For $f = u^* \star u$, the operator $\pi(f) = \pi(u)^*\pi(u)$ is positive:

$$\langle \pi(f)v, v\rangle = \langle \pi(u)v, \pi(u)v\rangle \geqslant 0, \qquad v \in V_\pi.$$

Plancherel theorem (1 of 2)

For $f \in C_c(G)$, the operators $\pi(f)$ for $\pi \in \widehat{G}$ are Hilbert–Schmidt. Moreover, there is a unique measure μ_{Pl} on $\widehat{G}$ such that

$$\int_G |f(g)|^2 \, \mathrm{d}g = \int_{\widehat{G}} \|\pi(f)\|_{\mathrm{HS}}^2 \, \mathrm{d}\mu_{\mathrm{Pl}}(\pi), \quad f \in C_c(G).$$

The support of μ_{Pl} is the tempered unitary dual $\widehat{G}_{\mathrm{temp}}$. The map $f \mapsto (\pi \mapsto \pi(f))$ extends to a unitary $G \times G$-equivariant map

$$L^2(G) \cong \int_{\widehat{G}} \mathrm{HS}(V_\pi) \, \mathrm{d}\mu_{\mathrm{Pl}}(\pi),$$

where the representation η_π of $G \times G$ on $\mathrm{HS}(V_\pi)$ is given by $\eta_\pi(x, y)(T) = \pi(x) T \pi(y^{-1})$.

Plancherel theorem (2 of 2)

For $f \in C_c^\infty(G)$, the operators $\pi(f)$ for $\pi \in \widehat{G}$ are of trace class, and the following inversion formula holds:

$$f(g) = \int_{\widehat{G}} \mathrm{tr}\left(\pi(f)\pi(g^{-1})\right) \mathrm{d}\mu_{\mathrm{Pl}}(\pi), \quad f \in C_c^\infty(G).$$

Here one can restrict to $g = 1$ without any loss of generality. Then, the inversion formula follows from the Plancherel identity and the theorem of Dixmier–Malliavin (1978): $C_c^\infty(G)$ is spanned by the functions $f = u^* \star u$ for $u \in C_c^\infty(G)$.

These formulae hold in great generality (see next slide). For the case at hand, $G = \mathrm{SL}_2(\mathbb{C})$, they were proved by Gelfand–Naimark (1947 & 1950). In this case, for an appropriate Haar measure $\mathrm{d}g$ on G, the Plancherel measure on $\widehat{G}$ can be described explicitly as

$$\mathrm{d}\mu_{\mathrm{Pl}}(\pi_{it,p}) = (t^2 + p^2)\,\mathrm{d}t\,\mathrm{d}p,$$

where $\mathrm{d}t$ is the Lebesgue measure on $\mathbb{R}_{\geqslant 0}$, and $\mathrm{d}p$ is the counting measure on $\frac{1}{2}\mathbb{Z}$.

Historical remarks on the Plancherel theorem

The above results hold almost verbatim for any second countable, unimodular, locally compact, type I group G. This was proved independently by Mautner (1950) and Segal (1950), based on the work of Murray–Neumann (1936) and Neumann (1949). They are the synthesis of several earlier major developments in number theory, group theory, analysis, and the theory of operator algebras.

- circle: Parseval (1799), Fourier (1807), Riesz–Fischer (1907)
- finite abelian groups: Gauss (1801), Dirichlet (1837)
- finite groups: Frobenius (1896), Burnside (1904), Schur (1905), Noether (1925)
- compact groups: Hurwitz (1897), Schur (1924), Weyl (1926), Peter–Weyl (1927), Haar (1933)
- real line: Plancherel (1910), Riesz (1910)
- locally compact abelian groups: Pontryagin (1934), van Kampen (1935), Weil (1940)
- semisimple Lie groups: Gelfand–Naimark (1947 & 1950), Gelfand–Graev (1953), Harish–Chandra (1951–1976)

Bounding the sup-norm via an automorphic kernel (1 of 2)

Recall the notations of the main theorem. Following Selberg (1956), consider a rapidly decaying continuous function $f \in L^1(G)$, and its action on $L^2(\Gamma\backslash G)$ given by

$$(R(f)\psi)(g) := \int_G f(h)\psi(gh)\,\mathrm{d}h = \int_{\Gamma\backslash G}\left(\sum_{\gamma\in\Gamma} f(g^{-1}\gamma h)\right)\psi(h)\,\mathrm{d}h.$$

Assume that $R(f)$ is a positive operator, and $\pi(f)$ acts by a scalar $c(\pi,\ell)$ on V_π^ℓ. Then $R(f)$ preserves the orthogonal decomposition $V_\pi^\ell \oplus V_\pi^{\ell,\perp}$. Moreover, $R(f)$ composed with the projection to V_π^ℓ has a simple kernel just like $R(f)$:

$$(R(f)_\pi^\ell\psi)(g) = \int_{\Gamma\backslash G}\left(c(\pi,\ell)\sum_{|q|\leqslant\ell}\phi_q(g)\overline{\phi_q(h)}\right)\psi(h)\,\mathrm{d}h.$$

Bounding the sup-norm via an automorphic kernel (2 of 2)

By a simple approximation argument, on the diagonal $g = h$, the kernel of $R(f)_\pi^\ell$ is upper bounded by the kernel of $R(f)$:

$$c(\pi, \ell) \sum_{|q| \leqslant \ell} |\phi_q(g)|^2 \leqslant \sum_{\gamma \in \Gamma} f(g^{-1}\gamma g), \quad g \in G.$$

To make this work in practice, we consider all functions $f \in L^2(G)$ for which every $\pi_{it,p}(f)$ acts by a scalar on the component $V_{it,p}^\ell$, and by zero on the other components $V_{it,p}^m$. These functions form a Hilbert subspace $\mathcal{H}(\tau_\ell) \subset L^2(G)$ defined by the conditions

- $f(g) = f(kgk^{-1})$ for almost every $g \in G$ and $k \in K$;
- $f = \overline{\chi}_\ell \star f \star \overline{\chi}_\ell$, where χ_ℓ is $2\ell + 1$ times the character of τ_ℓ.

For $f \in \mathcal{H}(\tau_\ell)$ rapidly decaying and continuous, the scalar $c(\pi, \ell)$ exists, and assuming $\pi \cong \pi_{it,p}$ it equals $\hat{f}(it, p)/(2\ell + 1)$, where

$$\widehat{f}(it, p) := \mathrm{tr}\big(\pi_{it,p}(f)\big) = \int_G f(g)\, \varphi_{it,p}^\ell(g)\, \mathrm{d}g,$$

$$\varphi_{it,p}^\ell(g) := \mathrm{tr}\big(\pi_{it,p}(\overline{\chi}_\ell)\pi_{it,p}(g)\pi_{it,p}(\overline{\chi}_\ell)\big).$$

Generalized spherical transform

The Plancherel theorem yields readily the Hilbert space isomorphism $\mathcal{H}(\tau_\ell) \cong L^2(\widehat{G}(\tau_\ell))$ with the Plancherel identity

$$\int_G |f(g)|^2 \,\mathrm{d}g = \frac{1}{2\ell+1} \sum_{\substack{|p|\leqslant\ell \\ p\equiv\ell \bmod 1}} \int_0^\infty |\widehat{f}(it,p)|^2 \,(t^2+p^2)\,\mathrm{d}t.$$

Moreover, for $f \in C_c^\infty(G) \cap \mathcal{H}(\tau_\ell)$, we have the inversion

$$f(g) = \frac{1}{2\ell+1} \sum_{\substack{|p|\leqslant\ell \\ p\equiv\ell \bmod 1}} \int_0^\infty \widehat{f}(it,p)\,\varphi^\ell_{it,p}(g^{-1})\,(t^2+p^2)\,\mathrm{d}t.$$

In practice we define $f(g)$ in terms of its generalized spherical transform $\widehat{f}(it,p)$. We need to ensure that $f(g)$ is continuous, rapidly decaying, and of reasonable size. For this we need to understand the spherical trace function $\varphi^\ell_{it,p}(g)$ in some detail.

Spherical trace function

The spherical trace function has an integral representation over K involving the diagonal matrix coefficients of τ_ℓ. As a result, it extends holomorphically to $(\nu, p, g) \in \mathbb{C} \times \frac{1}{2}\mathbb{Z} \times G$, and it satisfies a soft general bound that we skip for simplicity. In particular,

$$\varphi^\ell_{\nu,\ell}(g) = (2\ell+1) \int_K \kappa_\ell(k^{-1}gk)\,\mathrm{d}k,$$

where

$$\kappa_\ell\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := \bar{a}^{2\ell}\left(|a|^2+|c|^2\right)^{\nu-\ell-1}.$$

The spherical trace function also has remarkable symmetries:

$$\varphi^\ell_{\nu,p}(g) = \overline{\varphi^\ell_{-\overline{\nu},p}(g)} = \varphi^\ell_{\nu,p}(g^{-1}),$$

$$\varphi^\ell_{\nu,p}(g) = \varphi^\ell_{p,\nu}(g), \quad \nu \equiv p \pmod 1, \quad |\nu|, |p| \leqslant \ell.$$

These properties become transparent by analytically continuing the representations $\pi_{it,p}$ to (non-unitary Frechét) representation $\pi_{\nu,p}$.

Generalized principal series (1 of 2)

For $(\nu, p) \in \mathbb{C} \times \frac{1}{2}\mathbb{Z}$, let $V_{\nu,p}$ be the space of functions $v : \mathbb{C}^2 \to \mathbb{C}$ that are infinitely many times differentiable on $\mathbb{C}^2 \setminus \{(0,0)\}$ with respect to both variables and their conjugates, and satisfy

$$v(\lambda z_1, \lambda z_2) = |\lambda|^{2\nu-2}(\lambda/|\lambda|)^{-2p} v(z_1, z_2), \qquad \lambda \in \mathbb{C}^\times.$$

A sequence of functions is said to converge to zero if, on every compact subset of $\mathbb{C}^2 \setminus \{(0,0)\}$, they converge uniformly to zero together with all their derivatives. This makes $V_{\nu,p}$ into a Fréchet space. The action of $G = \mathrm{SL}_2(\mathbb{C})$ on $V_{\nu,p}$ is given by

$$\pi_{\nu,p}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) v(z_1, z_2) := v(az_1 + cz_2, bz_1 + dz_2).$$

This space has an invariant Hermitian inner product if and only if:

- $\nu \in i\mathbb{R}$; $\rightsquigarrow$ principal series $\rightsquigarrow$ tempered unitary dual $\widehat{G}_{\mathrm{temp}}$
- or $p = 0$ and $\nu \in (-1, 0) \cup (0, 1)$. $\rightsquigarrow$ complementary series

Generalized principal series (2 of 2)

Using the action of $K = \mathrm{SU}_2(\mathbb{C})$ and its diagonal subgroup $\left\{\mathrm{diag}(e^{i\varrho}, e^{-i\varrho}) : \varrho \in \mathbb{R}\right\}$, we can decompose the K-finite part of each $V_{\nu,p}$ into finite-dimensional subspaces and further into one-dimensional subspaces:

$$V_{\nu,p}^{K\text{-finite}} = \bigoplus_{\substack{\ell \geqslant |p| \\ \ell \equiv p \bmod 1}} V_{\nu,p}^{\ell} = \bigoplus_{\substack{\ell \geqslant |p| \\ \ell \equiv p \bmod 1}} \bigoplus_{\substack{|q| \leqslant \ell \\ q \equiv \ell \bmod 1}} V_{\nu,p}^{\ell,q}$$

- If $\nu \not\equiv p \bmod 1$ or $|\nu| \leqslant |p|$, then $\pi_{\nu,p} \cong \pi_{-\nu,-p}$ is irreducible.
- If $\nu \equiv p \bmod 1$ and $|\nu| > |p|$, then $\pi_{\nu,p}$ and $\pi_{-\nu,-p}$ are reducible. Assume $\nu > 0$, say. Then the sum of $V_{\nu,p}^{\ell}$ with $|p| \leqslant \ell < \nu$ is a closed invariant subspace of $V_{\nu,p}$, and the representation induced on the quotient is irreducible. The closure of the sum of $V_{-\nu,-p}^{\ell}$ with $\ell \geqslant \nu$ is an invariant subspace of $V_{-\nu,-p}$, and the representation induced on it is irreducible. Both of these representations of G are isomorphic to $\pi_{p,\nu} \cong \pi_{-p,-\nu}$.

Paley–Wiener space and Schwartz space

Now we see that if $f \in L^1(G) \cap \mathcal{H}(\tau_\ell)$ decays rapidly, then its transform $\widehat{f}$ extends holomorphically to $\mathbb{C} \times \frac{1}{2}\mathbb{Z}$ such that

$$\widehat{f}(\nu, p) = \widehat{f}(p, \nu), \quad \nu \equiv p \pmod 1, \quad |\nu|, |p| \leqslant \ell. \qquad (*)$$

This means that ν and p are not independent as we thought!

Theorem (Wang 1974)

For $f \in \mathcal{H}(\tau_\ell)$ and $R > 0$, the following conditions are equivalent.

1. *f is smooth, and $f(k_1 a_h k_2) = 0$ for $|h| > R$ and $k_1, k_2 \in K$.*
2. *$\widehat{f}$ extends holomorphically to $\mathbb{C} \times \frac{1}{2}\mathbb{Z}$ such that $(*)$ holds true, and we also have $\widehat{f}(\nu, p) \ll_C (1 + |\nu|)^{-C} e^{R|\Re\nu|}$.*

Theorem (Blomer–Harcos–Maga–Milićević 2021)

For $f \in \mathcal{H}(\tau_\ell)$, the following conditions are equivalent.

1. *f is smooth, and $\frac{\partial^m}{\partial h^m} f(k_1 a_h k_2) \ll_{m,A} e^{-A|h|}$ for $k_1, k_2 \in K$.*
2. *$\widehat{f}$ extends holomorphically to $\mathbb{C} \times \frac{1}{2}\mathbb{Z}$ such that $(*)$ holds true, and we also have $\widehat{f}(\nu, p) \ll_{B,C} (1 + |\nu|)^{-C}$ for $|\Re\nu| \leqslant B$.*

Choice of test function

We ended up using the function $f \in \mathcal{H}(\tau_\ell)$ whose transform equals

$$\widehat{f}(\nu, p) = \begin{cases} e^{(p^2-\ell^2+\nu^2)/2}, & \nu \in \mathbb{C}, \quad p \in \frac{1}{2}\mathbb{Z}, \quad |p| \leqslant \ell; \\ 0, & \nu \in \mathbb{C}, \quad p \in \frac{1}{2}\mathbb{Z}, \quad |p| > \ell. \end{cases}$$

This provides a positive operator $R(f)$ on $L^2(\Gamma\backslash G)$ such that

$$c(\pi, \ell) \gg 1/\ell \qquad \text{and} \qquad f(g) \ll \ell^2 e^{-\log^2 \|g\|}.$$

However, this only yields the baseline bound

$$\sum_{|q| \leqslant \ell} |\phi_q(g)|^2 \ll \ell^3.$$

In order to improve on this, we need to amplify π by Hecke operators. This idea was introduced by Iwaniec–Sarnak (1995).