Beyond the spherical sup-norm problem III

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Positivity

Let A be a positive operator on $L^2(\Gamma\backslash G)$, and assume it acts on a finite orthonormal set $\mathcal{B}$ of eigenfunctions with eigenvalues $(c_\phi(A))_{\phi\in\mathcal{B}}$. For any $\psi \in L^2(\Gamma\backslash G)$, let $\psi_{\mathcal{B}^\perp} = \mathrm{pr}_{\mathcal{B}^\perp}(\psi)$ and $\psi_{\mathcal{B}} = \psi - \psi_{\mathcal{B}^\perp}$. Then

$$\begin{aligned} \langle A\psi, \psi\rangle &= \langle A\psi_{\mathcal{B}}, \psi_{\mathcal{B}}\rangle + \langle A\psi_{\mathcal{B}^\perp}, \psi_{\mathcal{B}^\perp}\rangle \\ &\geq \langle A\psi_{\mathcal{B}}, \psi_{\mathcal{B}}\rangle = \sum_{\phi\in\mathcal{B}} c_\phi(A)|\langle\psi,\phi\rangle|^2. \end{aligned}$$

We will construct A in the form $A = R(f)R_{\mathrm{fin}}(\mathbf{x})$ where $R(f)$ and $R_{\mathrm{fin}}(\mathbf{x})$ are commuting and individually positive operators. Here, $R(f)$ is the convolution operator presented by Gergő coming from a kernel function f, and

$$R_{\mathrm{fin}}(\mathbf{x}) = \sum_{n\in\mathbb{Z}[i]\backslash\{0\}} x_n T_n,$$

where the T_n's are Hecke operators, and the finitely supported coefficient sequence $\mathbf{x} = (x_n)_{n\in\mathbb{Z}[i]}$ will guarantee that $R_{\mathrm{fin}}(\mathbf{x})$ is also positive.

Hecke operators

Denote by Γ_n the set of Gauss-integral 2x2 matrices of determinant n. For $\gamma \in \Gamma_n$, let $\tilde{\gamma} = \gamma/\sqrt{n}$ with an arbitrary choice of the square-root. For any $n \in \mathbb{Z}[i] \setminus \{0\}$, the Hecke operator T_n on $L^2(\Gamma\backslash G)$ is defined as ($\psi \in L^2(\Gamma\backslash G)$):

$$(T_n\psi)(g) = \frac{1}{|n|} \sum_{\gamma \in \Gamma\backslash\Gamma_n} \psi(\tilde{\gamma} g)$$
$$= \frac{1}{4|n|} \sum_{\substack{ad=n \\ b \bmod d}} \psi\left(\frac{1}{\sqrt{n}} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right).$$

The Hecke operators are self-adjoint and form a commuting family, more concretely, the product of any two is computed as

$$T_m T_n = \sum_{(d)|(m,n)} T_{mn/d^2}.$$

They also commute with the $R(f)$.

Achieving positivity for $R_{\mathrm{fin}}(\mathbf{x})$

Let $P \subset \mathbb{Z}[i] \setminus \{0\}$ be a finite set of primes. Since the T_n's themselves are self-adjoint, for any sequences $(y_l)_{l\in P}, (z_l)_{l\in P}$ of complex numbers, the operator

$$R_{\mathrm{fin}}(\mathbf{x}) = \left(\sum_{l\in P} y_l T_l\right) \cdot \left(\sum_{m\in P} \overline{y_m} T_m\right) + \left(\sum_{l\in P} z_l T_{l^2}\right) \cdot \left(\sum_{m\in P} \overline{z_m} T_{m^2}\right),$$

i.e.

$$x_n = \sum_{\substack{l,m\in P \\ (d)|(l,m) \\ n=lm/d^2}} y_l \overline{y_m} + \sum_{\substack{l,m\in P \\ (d)|(l^2,m^2) \\ n=l^2m^2/d^2}} z_l \overline{z_m}$$

is positive.

The amplified pre-trace inequality I

Recall from Gergő's talk, how $R(f)$ was defined:

$$(R(f)\psi)(g) = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(g^{-1}\gamma h)\psi(h)\,\mathrm{d}h.$$

Now apply this to $R_{\mathrm{fin}}(\mathbf{x})\psi = \sum_n x_n T_n \psi$, after some manipulation, we get that

$$\langle A\psi,\psi\rangle = \iint_{(\Gamma\backslash G)^2} \sum_n \frac{x_n}{|n|} \sum_{\gamma\in\Gamma_n} f(g^{-1}\tilde{\gamma}h)\psi(h)\overline{\psi(g)}\,\mathrm{d}h\,\mathrm{d}g.$$

Let V be the cuspidal component (of archimedean parameters ν, p) in which we want to estimate the automorphic forms. Assume that

$$R(f)|_{V_\ell} = \frac{\widehat{f}(V)}{2\ell+1}\cdot\mathrm{id} = \frac{\widehat{f}(\nu,p)}{2\ell+1}\cdot\mathrm{id},$$

$$R_{\mathrm{fin}}(\mathbf{x})|_V = \widehat{\mathbf{x}}(V)\cdot\mathrm{id} = \sum_n x_n\lambda_V(n)\cdot\mathrm{id},$$

where $\lambda_V(n)$ is the nth Hecke eigenvalue of V.

The amplified pre-trace inequality II

Now the positivity argument together with an approximation by ψ of the Dirac measure at a point Γg yield

$$\frac{\widehat{f}(V)\widehat{\mathbf{x}}(V)}{2\ell+1}\sum_{\phi\in\mathcal{B}}|\phi(g)|^2 \leq \sum_n \frac{x_n}{|n|}\sum_{\gamma\in\Gamma_n} f(g^{-1}\gamma g).$$

Analogously, we can focus on specific vectors in the Wigner basis by considering $R(f_q)$ in place of $R(f)$, with

$$f_q(g) := \frac{1}{2\pi}\int_0^{2\pi} f(g\mathrm{diag}(e^{i\varrho}, e^{-i\varrho}))e^{2qi\varrho}\,\mathrm{d}\varrho.$$

Then $R(f_q)$ projects the τ_ℓ-isotypical component to the one-dimensional subspace of functions which are transformed as $\psi(g\mathrm{diag}(e^{i\varrho}, e^{-i\varrho})) = \psi(g)e^{2qi\varrho}$.

A similar argument yields then

$$\frac{\widehat{f}(V)\widehat{\mathbf{x}}(V)}{2\ell+1}|\phi_q(g)|^2 \leq \sum_n \frac{x_n}{|n|}\sum_{\gamma\in\Gamma_n} f_q(g^{-1}\gamma g).$$

Choice of the amplifier

Let L be a parameter (to be chosen at the very end). Let $P(L)$ be the set of primes in $\mathbb{Z}[i]$ of argument between 0 and $\pi/4$, and norm between L and $2L$, then $P(L) \neq \emptyset$ for $L \geq 7$. Let $y_l = \operatorname{sgn}(\lambda_l(V))$ and $z_l = \operatorname{sgn}(\lambda_{l^2}(V))$ for $l \in P(L)$. Then

$$x_n = \begin{cases} \sum_{l \in P(L)} (y_l^2 + z_l^2) \ll L/\log L, & \text{if } n = 1, \\ (1 + \delta_{l_1 \neq l_2}) y_{l_1} y_{l_2} + \delta_{l_1 = l_2} z_{l_1} z_{l_2} \ll 1, & \text{if } n = l_1 l_2 \text{ for some } l_1, l_2 \in P(L), \\ (1 + \delta_{l_1 \neq l_2}) z_{l_1} z_{l_2} \ll 1, & \text{if } n = l_1^2 l_2^2 \text{ for some } l_1, l_2 \in P(L), \\ 0, & \text{otherwise.} \end{cases}$$

Also, since $T_1 = \mathrm{id} = T_l T_l - T_{l^2}$ for any prime l, $\max(|\lambda_l(V)|, |\lambda_{l^2}(V)|) \geq 1/2$, hence

$$\widehat{\mathbf{x}}(V) = \left(\sum_{l \in P(L)} |\lambda_l(V)| \right)^2 + \left(\sum_{l \in P(L)} |\lambda_{l^2}(V)| \right)^2 \gg \frac{L^2}{\log^2 L}.$$

Setting up the task of estimating generalized spherical functions and counting matrices I

This choice leads to

$$\frac{L^{2-\varepsilon}}{\ell}\sum_{\phi\in\mathcal{B}}|\phi(g)|^2 \ll_{\varepsilon,I} \sum_{n}\frac{|x_n|}{|n|}\sum_{\gamma\in\Gamma_n}|f(g^{-1}\tilde{\gamma}g)|.$$

We have the elementary counting, for $R>1$,

$$\#\{\gamma\in\Gamma_n : \|g^{-1}\tilde{\gamma}g\|\le R\}\ll_{\varepsilon,\Omega} R^{4+\varepsilon}|n|^{2+\varepsilon}.$$

[To get an idea why this is true, think of $g=\mathrm{id}$, then the norm conidition is $(a^2+b^2+c^2+d^2)/n\le R^2$. There are $O(R^4n^2)$ choices for a,d, then the number of solutions to $ad-bc=n$ can be estimated by the divisor bound (unless $ad=n$, but then we start from counting b,c).]

Recalling $f(g)\ll \ell^2 e^{-\log^2\|g\|}$, and splitting into dyadic ranges, we get that the contribution of large $\|g^{-1}\tilde{\gamma}g\|$ is small.

Setting up the task of estimating generalized spherical functions and counting matrices II

Recall also that f is the inverse transform of $\widehat{f}$, and the explicit inverse transform formula gives

$$f(g) \ll \ell \sup_{\nu \in i\mathbb{R}} |\varphi_{\nu,\ell}^{\ell}(g)| + \ell^{-50}.$$

Collecting these, we arrive at

$$\sum_{\phi \in \mathcal{B}} |\phi(g)|^2 \ll_{\varepsilon, I, \Omega} L^{-2+\varepsilon} \ell^2 \sum_{\substack{n, \gamma \in \Gamma_n \\ \log \|g^{-1}\tilde{\gamma} g\| \leq 8\sqrt{\log \ell}}} \frac{|x_n|}{|n|} \sup_{\nu \in i\mathbb{R}} |\varphi_{\nu,\ell}^{\ell}(g^{-1}\tilde{\gamma} g)| + L^{2+\varepsilon} \ell^{-48}$$

and its sibling

$$|\phi_q(g)|^2 \ll_{\varepsilon, I, \Omega} L^{-2+\varepsilon} \ell^2 \sum_{\substack{n, \gamma \in \Gamma_n \\ \log \|g^{-1}\tilde{\gamma} g\| \leq 8\sqrt{\log \ell}}} \frac{|x_n|}{|n|} \sup_{\nu \in i\mathbb{R}} |\varphi_{\nu,\ell}^{\ell,q}(g^{-1}\tilde{\gamma} g)| + L^{2+\varepsilon} \ell^{-48}.$$

Bounds on $\varphi^{\ell}_{\nu,\ell}$

Recall the formula

$$\varphi^{\ell}_{\nu,\ell}(g) := (2\ell+1)\int_K \kappa_\ell(k^{-1}gk)\,\mathrm{d}k,$$

where

$$\kappa_\ell\left(\begin{pmatrix} a & * \\ c & * \end{pmatrix}\right) = \overline{a}^{2\ell}(|a|^2+|c|^2)^{\nu-\ell-1}.$$

Obviously, $\varphi^{\ell}_{\nu,\ell}$ is invariant under K-conjugations, hence we can assume that g is upper triangular, i.e. $g = \left(\begin{smallmatrix} z & u \\ & z^{-1} \end{smallmatrix}\right)$ for some $z \in \mathbb{C}^\times$ and $u \in \mathbb{C}$.

Theorem (Blomer, Harcos, M., Milićević)

For $\ell \geq 1$, we have

$$\varphi^{\ell}_{\nu,\ell}(g) \ll_\varepsilon \min\left(\ell, \frac{\ell^\varepsilon \|g\|^6}{|z^2-1|^2}, \frac{\ell^{1/2+\varepsilon}\|g\|^3}{|u|}\right).$$

Iwasawa decomposition and simplifications

The bound

$$\varphi^{\ell}_{\nu,\ell}(g) \leq 2\ell + 1$$

follows from that $\varphi^{\ell}_{\nu,\ell}$ is the restricted trace of the unitary action $\pi_{\nu,p}(g)$ on a $2\ell+1$-dimensional space. As for the rest, after some manipulation, it suffices to prove that

$$\int_K \kappa_\ell(K(k^{-1}gk))\,\mathrm{d}k \ll_\varepsilon \ell^\varepsilon \min\left(\frac{\|g\|^4}{|z^2-1|^2\ell}, \frac{\|g\|}{|u|\sqrt{\ell}}\right),$$

where $K(g)$ is the K-part of g in the Iwasawa decomposition $G = KAN$, more concretely,

$$K\left(\begin{pmatrix} a & * \\ c & * \end{pmatrix}\right) = \begin{pmatrix} a/\sqrt{|a|^2+|c|^2} & * \\ * & * \end{pmatrix}.$$

Euler angles

It is convenient to parametrize K as

$$k(u,v,w)=\begin{pmatrix} e^{iu} & \\ & e^{-iu}\end{pmatrix}\begin{pmatrix}\cos v & i\sin v\\ i\sin v & \cos v\end{pmatrix}\begin{pmatrix} e^{iw} & \\ & e^{-iw}\end{pmatrix},$$

where $u\in[0,\pi)$, $v\in[0,\pi/2]$, $w\in[-\pi,\pi)$. This parametrization is essentially one-layer, except for those points with v-coordinate being an integer multiple of $\pi/2$. The Haar measure can be expressed then as

$$\mathrm{dk}=\frac{1}{4\pi^2}\sin 2v\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w.$$

(Sketch of the) proof of the claimed bound I

With the notation

$$x = (z^2 - 1)\cos\theta + ie^{-2i\phi}uz\sin\theta,$$

it suffices to prove that

$$\int_0^{\pi}\int_0^{\pi/2}\int_{-\pi}^{\pi}\left(\frac{1+\overline{x}\cos\theta}{\sqrt{|1+\overline{x}\cos\theta|^2+|\overline{x}\sin\theta|^2}}\right)^{2\ell}\mathrm{d}\psi\,\mathrm{d}\theta\,\mathrm{d}\phi$$
$$\ll_\varepsilon \ell^\varepsilon \min\left(\frac{\|g\|^4}{|z^2-1|^2\ell},\frac{\|g\|}{|u|\sqrt{\ell}}\right).$$

Note that

$$|z|, |z|^{-1}, |u| \leq \|g\|.$$

(Sketch of the) proof of the claimed bound II

Introduce the notation $\lambda = \sqrt{\log \ell}$. If

$$\tan\theta, |x| \sin\theta > \frac{100\lambda}{\sqrt{\ell}},$$

then the contribution is

$$\ll \left(1 - \frac{\log \ell}{\ell}\right)^{\ell} < \frac{1}{\ell},$$

which is admissible.
If $\tan\theta \leq 100\lambda/\sqrt{\ell}$, then the measure of set of θ's in the game is $O(\lambda/\sqrt{\ell})$, and because of the factor $\sin 2\theta = O(\lambda/\sqrt{\ell})$, the contribution is $O(\lambda^2/\ell)$, which is also admissible.

(Sketch of the) proof of the claimed bound III

If $|x|\sin\theta \leq 100\lambda/\sqrt{\ell}$, then we decompose the set of possible θ's as into $I(m,n)$ defined via $\sin\theta \asymp 2^{-m}$, $\cos\theta \asymp 2^{-n}$. The contribution of $\max(m,n) > 2\log\ell$ is admissible (it implies $\sin 2\theta \ll 1/\ell$). Then it suffices to prove the claimed bound for a fixed pair m, n. We may assume that $\min(m,n) = 0$.

There are two cases. **Either** both terms in

$$x = (z^2 - 1)\cos\theta + ie^{-2i\phi}uz\sin\theta$$

are $O(2^m\lambda/\sqrt{\ell})$ (with some fixed implied constant). Then the integrand together with the localization of θ is as small as promised:

$$\iiint \sin 2\theta \,\mathrm{d}\psi\,\mathrm{d}\theta\,\mathrm{d}\phi \ll 2^{-2\max(m,n)},$$

which is bounded as needed by the size conditions on the terms.

(Sketch of the) proof of the claimed bound IV

Or the two terms are individually large, but then their moduli must essentially be the same, and their angles must be essentially opposite of each other.
The moduli condition localizes θ to a set of measure

$$O(2^m\lambda/|uz|\sqrt{\lambda}) \qquad (\text{if } \sin\theta \leq \cos\theta)$$

or a set of measure

$$O(\lambda/|z^2-1|\sqrt{\ell}) \qquad (\text{if } \sin\theta \geq \cos\theta).$$

The angle condition localizes ϕ to a set of measure

$$O(2^{2m}\lambda/|uz|\sqrt{\ell}).$$

Collecting everything, we arrive at admissible bounds both for $\sin\theta \leq \cos\theta$ and $\sin\theta \geq \cos\theta$.

Notations

We introduce the following notations:

$$D(L,\mathcal{L}) = \{n \in \mathbb{Z}[i] : \mathcal{L} \le |n|^2 \le 16\mathcal{L},\ n = 1 \text{ or } n = l_1 l_2 \text{ or } n = l_1^2 l_2^2 \text{ for some } l_1, l_2 \in P(L)\}$$

and for $\delta = (\delta_1, \delta_2) \in \mathbb{R}^2_{>0}$,

$$M(g,L,\mathcal{L},\delta) = \sum_{n \in D(L,\mathcal{L})} \#\{\gamma \in \Gamma_n : g^{-1}\tilde{\gamma} g = k \begin{pmatrix} z & u \\ & z^{-1} \end{pmatrix} k^{-1} \text{ for some } k \in K,\ |z| \ge 1,\ \min|z \pm 1| \le \delta_1,\ |u| \le \delta_2\}.$$

The bounds on the x_n's and those on $\varphi^{\ell}_{\nu,\ell}$ altogether give

$$\sum_{\phi \in \mathcal{B}} |\phi(g)|^2 \ll_{\varepsilon, I, \Omega} \ell^{3+\varepsilon} L^{\varepsilon} \sum_{\substack{\delta \text{ dyadic} \\ 1/\sqrt{\ell} \le \delta_j \le \ell^{\varepsilon}}} \min\left(\frac{1}{\ell \delta_1^2}, \frac{1}{\sqrt{\ell}\delta_2}\right) \cdot \left(\frac{M(g,L,1,\delta)}{L} + \frac{M(g,L,L^2,\delta)}{L^3} + \frac{M(g,L,L^4,\delta)}{L^4}\right) + L^{2+\varepsilon}\ell^{-48}.$$

Counting preliminaries I

We count then Gauss-integral matrices $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of determinant n with

$$g^{-1}\tilde{\gamma}g = k\begin{pmatrix} z & u \\ & z^{-1} \end{pmatrix}k^{-1}, k \in K, |z| \geq 1, \min(|z \pm 1|) \leq \delta_1, |u| \leq \delta_2.$$

By doubing (taking $-\gamma$ for γ if needed), we may assume that it is the $+$ in the minimum, i.e. $|z-1|, |z^{-1}-1| \leq \delta_1$, which implies

$$\left|\frac{a+d}{\sqrt{n}} - 2\right| = |\mathrm{tr}(\tilde{\gamma}) - 2| = |z + z^{-1} - 2| = |z-1||z^{-1}-1| \leq \delta_1^2.$$

Also, since $g \in \Omega$,

$$\|\tilde{\gamma} - \mathrm{id}\| = \left\| gk\begin{pmatrix} z-1 & u \\ & z^{-1}-1 \end{pmatrix}k^{-1}g^{-1} \right\| \ll_\Omega \delta_1 + \delta_2.$$

Counting preliminaries II

Then (from now, not indicating the ε, Ω, I-dependence)

$$|a+d-2\sqrt{n}| \leq \delta_1^2\sqrt{|n|}, \qquad a-d, b, c \ll (\delta_1+\delta_2)\sqrt{|n|}.$$

This, introducing the notation $A \preceq B$ for $A \ll B\ell^\varepsilon L^\varepsilon$, implies $|a+d| \preceq \sqrt{|n|}$, and then

$$(a-d)^2+4bc = (a+d)^2-4n \preceq \delta_1^2|n|.$$

It will be useful, in some cases, to count separately the parabolic and non-parabolic matrices.

Countings I

Note that all along $|n| \asymp \mathcal{L}^{1/2}$.

Determinant one

We have $M(g, L, 1, \delta) \preceq 1$.

This is immediate from

$$|a + d - 2\sqrt{n}| \leq \delta_1^2 \sqrt{|n|}, \qquad a - d, b, c \ll (\delta_1 + \delta_2)\sqrt{|n|},$$

and that $\delta_1, \delta_2 \preceq 1$.
On this point, it is useful to note that the baseline bound $\ll \ell^3$ follows even on this point.

Countings II

Parabolic matrices

We have $M^{\mathrm{p}}(g, L, \mathcal{L}, \delta) \preceq \mathcal{L}^{1/2} + \mathcal{L}\delta_2^2$.

In the parabolic case, the preliminary bounds hold in the stronger form

$$a + d = 2\sqrt{n}, \qquad a - d, b, c \ll \delta_2\sqrt{|n|}.$$

If $bc \neq 0$, then there are $O(\mathcal{L}^{1/2})$ choices for $a + d$, $O(\mathcal{L}^{1/2}\delta_2^2)$ choices for $a - d \neq 0$. Since $a + d$ determines n, bc is fixed, and the divisor bound implies a $\preceq 1$ multiplier on the number of choices. This is admissible.
If $bc = 0$, then there are $O(\mathcal{L}^{1/2})$ choices for $a = d = \sqrt{n}$, and $O(1 + \mathcal{L}^{1/2}\delta_2^2)$ choices for one of b and c (the other is zero). This is admissible again.

Countings III

Note that

$$(a-d)^2+4bc=(a+d)^2-4n \preceq \delta_1^2|n|$$

implies that if there are non-parabolic matrices at all, then $\mathcal{L}^{-1/4} \preceq \delta_1$. We assume this from now on.

Non-parabolic matrices

We have

$$M^{\mathrm{np}}(g,L,L^2,\delta) \preceq L^4\delta_1^4(\delta_1^2+\delta_2^2),$$
$$M^{\mathrm{np}}(g,L,L^4,\delta) \preceq L^6\delta_1^4(\delta_1^2+\delta_2^2).$$

Countings IV

If $bc \neq 0$, then

$$a - d, b, c \ll (\delta_1 + \delta_2)\sqrt{|n|}$$

gives $O(\mathcal{L}^{1/2}(\delta_1^2 + \delta_2^2))$ choices for $a - d$, then

$$(a-d)^2 + 4bc = (a+d)^2 - 4n \preceq \delta_1^2 |n|$$

gives $O(\mathcal{L}\delta_1^4)$ choices for bc, the divisor bound splits this to b, c on the cost of a multiplier $\preceq 1$.

If $bc = 0$, then first we choose b, c, there are $O(\mathcal{L}^{1/2}(\delta_1^2 + \delta_2^2))$ options, then $O(\mathcal{L}\delta_1^4)$ many choices for $a - d$. Altogether, there are $O(\mathcal{L}^{3/2}\delta_1^4(\delta_1^2 + \delta_2^2))$ choices for the triple $(a - d, b, c)$.

Using $a + d \preceq \mathcal{L}^{1/4}$, there are $O(\mathcal{L}^{1/2})$ choices for $a + d$, which is sufficient for the middle range ($\mathcal{L} = L^2$) bound.

In the high range ($\mathcal{L} = L^4$), observe that $a + d$ is determined up to a divisor count, since in this case, the determinant is the square $l_1^2 l_2^2$, i.e.

$$(a-d)^2 + 4bc = (a+d)^2 - 4n = (a+d+2l_1 l_2)(a+d-2l_1 l_2).$$

Summing up

Collecting everything, we get

$$\sum_{\phi \in \mathcal{B}} |\phi(g)|^2 \preceq \ell^3 \left(\frac{1}{L} + \frac{1}{\sqrt{\ell}} + \frac{L^2}{\ell} \right).$$

We optimize this by choosing $L \sim \ell^{1/3}$, which gives the promised power-saving $3 - 1/3 = 8/3$.

Beyond the spherical sup-norm problem IV

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Recall

Recall from the previous talks that we want to estimate automorphic forms among the following circumstances. Assume $\phi : \Gamma\backslash G \to \mathbb{C}$ is L^2-normalized Hecke cusp form, which generates, as a G-representation, a principal series representation of parameter (ν, p) with $\nu \in I$, and, as a K-representation, a $2\ell + 1 = 2|p| + 1$-dimensional irreducible unitary representation. Let $g \in \Omega$ for some $\Omega \subset \Gamma\backslash G$ compact, our goal is to estimate $|\phi(g)|$ by a power of ℓ with implied constants possibly depending on Ω, I.
Last time, we proved a bound for $\sum_{\phi\in\mathcal{B}} \|\phi(g)\|^2$ for any orthonormal basis of the $2\ell + 1$-dimensional K-representation. The goal for today is, for a very specific choice of $\mathcal{B}$, to obtain a good bound for individual forms ϕ. Our choice is the Wigner basis $\{\phi_q : q = -\ell, \ldots, \ell\}$, where ϕ_q spans that one-dimensional subspace $V^{\ell,q}$, whose elements ψ transform under the diagonal part of K as

$$\psi\left(g\begin{pmatrix} e^{i\varrho} & \\ & e^{-i\varrho}\end{pmatrix}\right) = e^{2qi\varrho}\psi(g).$$

The projector to $V^{\ell,q}$

To select the space $V^{\ell,q}$, instead of the earlier $R(f)$, we apply $R(f_q)$, where

$$f_q(g) := \int_0^{2\pi} f(g\,\mathrm{diag}(\mathrm{e}^{\mathrm{i}\varrho}, \mathrm{e}^{-\mathrm{i}\varrho}))e^{2qi\varrho}\,\mathrm{d}\varrho.$$

Then $R(f_q) = R(f)\Pi_q = \Pi_q R(f)$, where Π_q is the orthogonal projection of V^ℓ to $V^{\ell,q}$. The inversion formula looks as

$$f_q(g) = \frac{1}{2\ell+1}\sum_{|p|\le\ell}\int_0^\infty e^{(p^2-\ell^2-t^2)/2}\varphi_{it,p}^{\ell,q}(g^{-1})(t^2+p^2)\,\mathrm{d}t,$$

where

$$\varphi_{it,p}^{\ell,q}(g) = \frac{1}{2\pi}\int_0^{2\pi}\varphi_{\nu,p}^{\ell}(g\,\mathrm{diag}(e^{i\varrho}, e^{-i\varrho}))e^{-2qi\varrho}\,\mathrm{d}\varrho.$$

Again, $f_q(g)$ drops very fast as $\|g\|$ gets large, hence we have, just like last week,

$$|\phi_q(g)|^2 \ll_{\varepsilon,I,\Omega} L^{-2+\varepsilon}\ell^2 \sum_{\substack{n,\gamma\in\Gamma_n \\ \log\|g^{-1}\tilde{\gamma}g\|\le 8\sqrt{\log\ell}}} \frac{|x_n|}{|n|}\sup_{\nu\in i\mathbb{R}}|\varphi_{\nu,\ell}^{\ell,q}(g^{-1}\tilde{\gamma}g)| + L^{2+\varepsilon}\ell^{-48}.$$

Bounds on $\varphi_{\nu,\ell}^{\ell,q}$

Therefore, we need good bounds on $\varphi_{\nu,\ell}^{\ell,q}$. Denote by $\mathcal{D}$ the set of diagonal matrices.

Theorem (Blomer, Harcos, M., Milićević)

Let $\ell, q \in \mathbb{Z}$ such that $\ell \geq 1, |q|$. Then for any $\Lambda, \varepsilon > 0$, we have

$$\varphi_{\nu,\ell}^{\ell,q}(g) \ll_{\Lambda,\varepsilon} \ell^{\varepsilon} \min\left(1, \frac{\|g\|}{\sqrt{\ell}\mathrm{dist}(g,K)^2\mathrm{dist}(g,\mathcal{D})}\right) + \ell^{-\Lambda}.$$

Sketch of the proof I

We write g in Cartan coordinates, i.e.

$$g = k(u_1, v_1, w_1)\mathrm{diag}(r, r^{-1})k(u_2, v_2, w_2).$$

Then writing explicitly out the formula for $\varphi_{\nu,\ell}^{\ell,q}$ gives that up to constant, it is

$$(2\ell+1)\int_{\substack{0\le u\le\pi\\ 0\le v\le\pi/2\\ 0\le w\le 2\pi\\ 0\le\varrho\le 2\pi}} \kappa_\ell(k(-w,-v,-u)k(u_1,v_1,w_1)\mathrm{diag}(r,r^{-1})k(u_2,v_2,w_2)k(0,0,\varrho)k(u,v,w))\, e^{-2iq\varrho}\sin 2v \,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}\varrho,$$

where

$$\kappa_\ell\left(\begin{pmatrix} a & b\\ * & * \end{pmatrix}\right) = \overline{a^{2\ell}}(|a|^2+|c|^2)^{\nu-\ell-1}.$$

Sketch of the proof II

Changing the variables $k(u_2, v_2, w_2)k(0,0,\varrho)k(u,v,w) \mapsto k(u,v,w)$, dropping the irrelevant w-integration (which does not affect κ_ℓ), and changing again the variables $\varrho \mapsto \varrho - u_1 - w_2$, we see that the quantity in question is, up to constant and phase,

$$(2\ell+1)\int_{u,v,\varrho} \overline{e^{-i\varrho}I + e^{i\varrho}J}^{2\ell} e^{-2iq\varrho}(r^2\cos^2 v + r^{-2}\sin^2 v)^{\nu-\ell-1}\sin 2v \,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\varrho,$$

where

$$I := \left(r^{-1}e^{-2iu-iw_1}\sin v\cos v_1 + re^{iw_1}\cos v\sin v_1\right)\left(e^{2iu-iu_2}\sin v\cos v_2 - e^{iu_2}\cos v\sin v_2\right),$$
$$J := \left(-r^{-1}e^{-2iu-iw_1}\sin v\sin v_1 + re^{iw_1}\cos v\cos v_1\right)\left(e^{2iu-iu_2}\sin v\sin v_2 + e^{iu_2}\cos v\cos v_2\right).$$

On this point, we can evaluate the ϱ integral to get, up to constant and phase

$$(2\ell+1)\binom{2\ell}{\ell+q}\int_{u,v}\frac{\sin 2v}{(r^2\cos^2 v + r^{-2}\sin^2 v)^{\nu-\ell-1}}\,\bar{I}^{\ell+q}\bar{J}^{\ell-q}\,\mathrm{d}u\,\mathrm{d}v.$$

Sketch of the proof III

Setting now $t := r^{-1} \tan v$, $\phi := 2u$, massaging a little further, then changing $\phi \mapsto \phi + u_2 - w_1$ and setting $\Delta := u_2 + w_1$, we finally arrive at, up to constant and phase as usual,

$$(2\ell+1)\binom{2\ell}{\ell+q}\int_0^\infty \frac{t}{((1+(t/r)^2)(1+(tr)^2))^{\ell+1}} \times \int_0^{2\pi} \left|e^{i\phi+i\Delta} t\cos v_1 + \sin v_1\right|^{\ell+q}\left|e^{i\phi-i\Delta} t\cos v_2 - \sin v_2\right|^{\ell+q} \left|e^{i\phi+i\Delta} t\sin v_1 - \cos v_1\right|^{\ell-q}\left|e^{i\phi-i\Delta} t\sin v_2 + \cos v_2\right|^{\ell-q} \mathrm{d}\phi\,\mathrm{d}t.$$

Recall the earlier notation $\lambda = \sqrt{\log \ell}$.

Interlude: a Young-type inequality

Lemma

Let ℓ, q, Λ as in the Theorem. Let further $X > 0$. (a) If $A, B \geq 0$ satisfy $A^2 + B^2 = X^2$, then

$$\left(\frac{2\ell}{\ell+q}\right)^{(\ell+q)/2}\left(\frac{2\ell}{\ell-q}\right)^{(\ell-q)/2} A^{\ell+q}B^{\ell-q} \leq X^{2\ell}.$$

Moreover, the left-hand side is $O_\Lambda(X^{2\ell}\ell^{-\Lambda})$ unless

$$A^2 = \frac{\ell+q}{2\ell}X^2 + O_\Lambda\left(X^2\frac{\lambda^2+\lambda\sqrt{\ell-|q|}}{\ell}\right),$$

$$B^2 = \frac{\ell-q}{2\ell}X^2 + O_\Lambda\left(X^2\frac{\lambda^2+\lambda\sqrt{\ell-|q|}}{\ell}\right).$$

(b) If $A, B, C, D \geq 0$ satisfy $A^2 + B^2 = C^2 + D^2 = X^2$, then

$$\binom{2\ell}{\ell+q}A^{\ell+q}B^{\ell-q}C^{\ell+q}D^{\ell-q} \ll \frac{X^{4\ell}}{1+\sqrt{\ell-|q|}}.$$

Moreover, the left-hand side is $O_\Lambda(X^{4\ell}\ell^{-\Lambda})$ unless A, B are in the above-indicated domains, and the analogous estimates for C, D are satisfied.

Proof of the Young-type inequality I

Assume for simplicity that $|q| < \ell$. The first bound is exactly the Young inequality upon choosing

$$x := \left(\sqrt{\frac{2\ell}{\ell+q}}\frac{A}{X}\right)^{\frac{\ell+q}{\ell}}, \qquad a := \frac{2\ell}{\ell+q},$$

$$y := \left(\sqrt{\frac{2\ell}{\ell-q}}\frac{B}{X}\right)^{\frac{\ell-q}{\ell}}, \qquad b := \frac{2\ell}{\ell-q},$$

which in particular implies $1/a + 1/b = 1$.
Also, the left-hand side in the display in (a) is $O_\Lambda(X^{2\ell}\ell^{-\Lambda})$, unless

$$xy > 1/2, \qquad xy = 1 + O_\Lambda(\delta), \qquad \delta = \lambda^2/\ell.$$

This implies $1/3 < x, y < 3/2$ by $x^a/a + y^b/b = 1$.

Proof of the Young-type inequality II

Let, say $q \geq 0$, i.e. $a \leq b$, then $x^a < a \leq 2$. Also,

$$b \log x < (b/a) \log a < (b/a)(a-1) = 1,$$

which, from $xy = 1 + O_\Lambda(\delta)$, gives that $b \log y > -1 + O_\Lambda(b\delta)$, i.e. if $b\delta < 1$, then $y^b \gg_\Lambda 1$. Introduce the function

$$F(t) := \frac{x^a}{a} + \frac{t^b}{b} - xt.$$

At $t := y_0 := x^{a-1}$ both F and F' vanishes, therefore, from the Taylor expansion (with Lagrange remainder term) shows that

$$\delta \gg_\Lambda 1 - xy = F(y) \geq \frac{b-1}{2} \min(y_0^{b-2}, y^{b-2})(y - y_0)^2.$$

Here $y_0^{b-2} = x^{2-a} \gg 1$. Now if $y^b > 1$ or $b\delta < 1$, then $y^b \gg_\Lambda 1$, and then $y - y_0 = O_\Lambda(\sqrt{\delta/b})$.

Proof of the Young-type inequality III

Then we have the following two approximations on bxy:

$$bxy = bxy_0 + O_\Lambda(\sqrt{b\delta}) = bx^a + O_\Lambda(\sqrt{b\delta}),$$
$$bxy = b + O_\Lambda(b\delta) = (b-1)x^a + y^b + O_\Lambda(b\delta),$$

and comparing them, we obtain

$$x^a - y^b \ll_\Lambda b\delta + \sqrt{b\delta}.$$

This inequality is immediate in the complement case $y^b \leq 1$ and $b\delta \geq 1$. Writing back, this means

$$aA^2 - bB^2 = X^2(b\delta + \sqrt{b\delta}),$$

and solving the system coming from this and $A^2 + B^2 = X^2$, we obtain the statement. The case $|q| = \ell$ is trivial.
The claim (b) follows from (a) and Stirling's approximation on factorials in the binomials.

Sketch of the proof IV

Returning to the proof, this buys us strong localizations, i.e. the integral is negligible (admissibly) outside a set $\mathcal{M}$ of pairs (ϕ, t) given by

$$\min(t, t^{-1}) \ll_\Lambda \frac{\lambda}{(r-1)\sqrt{\ell}}$$

and

$$2t \sin 2v_1 \cos(\phi + \Delta) = (1 - t^2) \cos 2v_1 + \frac{q}{\ell}(1 + t^2) + O_\Lambda\left((1+t^2)\frac{\lambda^2 + \lambda\sqrt{\ell - |q|}}{\ell}\right),$$
$$2t \sin 2v_2 \cos(\phi - \Delta) = (t^2 - 1) \cos 2v_2 - \frac{q}{\ell}(1 + t^2) + O_\Lambda\left((1+t^2)\frac{\lambda^2 + \lambda\sqrt{\ell - |q|}}{\ell}\right).$$

Recall that we want to prove

$$\varphi_{\nu,\ell}^{\ell,q}(g) \ll_{\Lambda,\varepsilon} \ell^\varepsilon \min\left(1, \frac{\|g\|}{\sqrt{\ell}\mathrm{dist}(g,K)^2\mathrm{dist}(g,\mathcal{D})}\right) + \ell^{-\Lambda}.$$

Sketch of the proof V

The first one in the red set of equations provides an equation of the form

$$\mu t^2 - 2t\rho\cos(\phi+\Delta) + \frac{2q}{\ell} - \mu + O_\Lambda\left(\frac{\sigma}{\ell}\right) = 0,$$

where μ, ρ, σ are expressed in terms of ℓ, q, v_1. Going by cases according to the size of the discriminant of this quadratic (in t) and the relative size of q to ℓ, a fixed ϕ localizes t or a fixed t localizes ϕ to small sets. (In certain cases, we also need some dyadic localization of certain quantities, including the discriminant.) Applying Fubini, the integration domain is small in all cases, and the resulting integral is admissible for the bound ℓ^ε.

Sketch of the proof VI

Unless

$$\|g\| < \sqrt{\ell}\mathrm{dist}(g,K)^2\mathrm{dist}(g,\mathcal{D}),$$

then we are done, since the second bound is weaker. In the remaining case, either the integration domain $\mathcal{M}$ is void, or not, and in the second case, we can fix some $(\phi, t) \in \mathcal{M}$. Feeding it into red, and utilizing also blue, we can squeeze v_j close (in terms of t) to an integer multiple of $\pi/2$ (with parity depending only on the signature of q). Now t is ruled by blue, hence we finally obtain

$$\mathrm{dist}(g,\mathcal{D}) \ll_\Lambda \|g\| \left(\frac{\lambda}{\mathrm{dist}(g,K)\sqrt{\ell}} + \frac{\lambda + \sqrt{\ell - |q|}}{\sqrt{\ell}} \right).$$

Massaging this a little further (and using the assumption on $\|g\|$), we obtain the claimed bound.

The setup of the counting problem

As earlier, for $\delta = (\delta_1, \delta_2) \in \mathbb{R}^2_{>0}$, introduce

$$M'(g, L, \mathcal{L}, \delta) := \sum_{n \in D(L,\mathcal{L})} \# \left\{ \gamma \in \Gamma_n : \mathrm{dist}\left(g^{-1}\tilde{\gamma} g, K\right) \leq \delta_1,\ \mathrm{dist}(g^{-1}\tilde{\gamma} g, \mathcal{D}) \leq \delta_2 \right\},$$

and supressing the ε, Ω, l-dependence from the notation,

$$|\phi_q(g)|^2 \ll \ell^{2+\varepsilon} L^{\varepsilon} \sum_{\substack{\delta \text{ dyadic},\ \delta_j \leq \ell^{\varepsilon} \\ \delta_1^2 \delta_2 \geq 1/\sqrt{\ell}}} \frac{1}{\sqrt{\ell}\delta_1^2 \delta_2}$$
$$\times \left(\frac{M'(g, L, 1, \delta)}{L} + \frac{M'(g, L, L^2, \delta)}{L^3} + \frac{M'(g, L, L^4, \delta)}{L^4} \right) + L^{2+\varepsilon} \ell^{-48}.$$

Away from the diagonal – preparation I

For

$$g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \in G,$$

we have

$$g^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} g = \begin{pmatrix} \frac{a+d}{2} + L_1 & L_2 \\ L_3 & \frac{a+d}{2} - L_1 \end{pmatrix},$$

with

$$\begin{aligned} L_1 &= (a-d)\left(\tfrac{1}{2} + g_2 g_3\right) + b g_3 g_4 - c g_1 g_2, \\ L_2 &= (a-d) g_2 g_4 + b g_4^2 - c g_2^2, \\ L_3 &= -(a-d) g_1 g_3 - b g_3^2 + c g_1^2. \end{aligned}$$

Away from the diagonal – preparation II

If we consider this as a linear system with unknowns $a-d, b, c$, then fixing one of the unknowns, and solving the last two equations when L_2, L_3 are (close to) 0, we get linear expressions (with some error) for the other two unknowns. This is expressed as follows.

Lemma

For $a, b, c, d \in \mathbb{C}$ and $\Delta > 0$, if $L_2, L_3 \ll \Delta$, then

$$(a-d, b, c) = s(\lambda_1, \lambda_2, \lambda_3) + O(\Delta),$$

where $\lambda_{1,2,3} \ll 1$, and s is one of $a-d, b, c$.
Also, if $(a-d)^2 + 4bc = 0$, then $a-d, b, c \ll \Delta$.

We bound now, for $\mathcal{L} \in \{1, L^2, L^4\}$ as before,

$$M'_{\mathcal{D}}(g, L, \mathcal{L}, \varepsilon, \delta_2) := \sum_{n \in D(L,\mathcal{L})} \# \left\{ \gamma \in \Gamma_n : \|g^{-1}\tilde{\gamma}g\| \ll \ell^{\varepsilon}, \mathrm{dist}(g^{-1}\tilde{\gamma}g, \mathcal{D}) \leq \delta_2 \right\}.$$

Away from the diagonal – counting I

We obviosuly have

$$\|\gamma\| \preceq \mathcal{L}^{1/4}, \qquad L_2, L_3 \ll \mathcal{L}^{1/4}\delta_2.$$

The first bound gives immediately

$$M'_{\mathcal{D}}(g, L, 1, \varepsilon, \delta_2) \preceq 1.$$

Record for the rest, i.e. $\mathcal{L} \in \{L^2, L^4\}$,

$$(a-d)^2 + 4bc = (a+d)^2 - 4n.$$

Away from the diagonal – counting II

Lemma

We have

$$M'_{\mathcal{D}}(g, L, L^2, \varepsilon, \delta_2) \preceq L^2 + L^4\delta_2^4,$$
$$M'_{\mathcal{D}}(g, L, L^4, \varepsilon, \delta_2) \preceq L^2 + L^6\delta_2^4.$$

We can apply the preparational Lemma with $\Delta = \mathcal{L}^{1/4}\delta_2$. In the parabolic case $(a-d)^2 + 4bc = 0$, by the addition to the lemma, $a-d, b, c \ll \mathcal{L}^{1/4}\delta_2$. If $bc \neq 0$, then there are $\ll \mathcal{L}^{1/2}$ choices for $a+d = \pm 2\sqrt{n}$, also $\preceq \mathcal{L}^{1/2}\delta_2^2$ choices for $a-d \neq 0$. Then bc is fixed, and we finish by the divisor bound. If $bc = 0$, then $a = d = \pm\sqrt{n}$, hence there are $\ll \mathcal{L}^{1/2}$ choices for a, d, and for the nonzero one of b, c, $\preceq \mathcal{L}^{1/2}\delta_2^2$ choices. In any case, the contribution is admissible.

Away from the diagonal – counting III

In the non-parabolic case, we can choose the s of the preparational Lemma in $\preceq \mathcal{L}^{1/2}$ ways. It determines the other two component up to a $\Delta = \mathcal{L}^{1/4}\delta_2$ error, hence in total, we have $\preceq \mathcal{L}^{1/2}(1+\mathcal{L}^{1/4}\delta_2)^4 \ll \mathcal{L}^{1/2} + \mathcal{L}^{3/2}\delta_2^4$ choices for the triple $(a-d, b, c)$.

In the middle range $\mathcal{L} = L^2$, we have $\ll \mathcal{L}^{1/2}$ choices for $a+d$, while in the high range, we can again apply the divisor bound, using that $n = l_1^2 l_2^2$ is a square on

$$(a-d)^2 + 4bc = (a+d+2l_1l_2)(a+d-2l_1l_2),$$

to see it is determined up to an ε-power.

Summing up

The counting "away from the maximal compact" already appears in the spherical case, and we can borrow the counts of an earlier Blomer–Harcos–Milićević paper (in Duke, 2016). Together with those, we see that

$$\begin{aligned} &M'(g, L, 1, \delta) \preceq 1, \\ &M'(g, L, L^2, \delta) \preceq \min\left(L^2 + L^4\delta_1, L^2 + L^4\delta_2^4\right), \\ &M'(g, L, L^4, \delta) \preceq \min\left(L^3 + L^6\delta_1, L^2 + L^6\delta_2^4\right). \end{aligned}$$

Summing over the dyadic intervals in δ, we arrive at

$$|\phi_q(g)|^2 \preceq \ell^2 \left(\frac{1}{L} + \frac{L^2}{\ell^{2/9}}\right).$$

The optimal choice is $L \sim \ell^{2/27}$, which leads to

$$|\phi_q(g)| \preceq \ell^{26/27}.$$