A Burgess-like subconvex bound for twisted L-functions

Gergely Harcos

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The problem

g: primitive holomorphic or Maass cusp form
χ: primitive Dirichlet character of conductor q
s: point on critical line ($\Re s = \frac{1}{2}$)

$$L(s, g \otimes \chi) \overset{?}{\ll}_{s,g} q^{\frac{1}{2}-\delta}$$

Applications

- bounds for Fourier coefficients of half-integral weight holomorphic or Maass cusp forms
- equidistribution of lattice points on ellipsoids
- equidistribution of Heegner points
- equidistribution of closed geodesics on hyperbolic surfaces
- subconvexity problem of Rankin–Selberg L-functions
- equidistribution of incomplete Galois orbits of Heegner points

Evolution of results

g: primitive holomorphic or Maass cusp form
χ: primitive Dirichlet character of conductor q
s: point on critical line ($\Re s = \frac{1}{2}$)

- $\delta < \frac{1}{22}$ for g holomorphic of full level (Duke–Friedlander–Iwaniec, 1993)

- $\delta < \frac{1}{8}$ for g holomorphic (Bykovskiĭ, 1996)

- $\delta < \frac{1}{54}$ (Harcos, 2001)

- $\delta < \frac{1}{22}$ (Michel, 2002)

- $\delta < \frac{1-2\theta}{10+4\theta}$ under Hypothesis H_θ (Blomer, 2004)

- $\delta < \frac{1-2\theta}{8}$ under Hypothesis H_θ (Blomer–Harcos–Michel, 2004)

Hypothesis H_θ**.** For any cuspidal automorphic form π on GL_2 over $\mathbb{Q}$ with local Hecke parameters $\alpha_\pi^{(1)}(p)$, $\alpha_\pi^{(2)}(p)$ for $p < \infty$ and $\mu_\pi^{(1)}(\infty)$, $\mu_\pi^{(2)}(\infty)$, one has the bounds

$$|\alpha_\pi^{(j)}(p)| \leqslant p^\theta, \text{ if } \pi_p \text{ is unramified;}$$
$$|\Re \mu_\pi^{(j)}(\infty)| \leqslant \theta, \quad \text{if } \pi_\infty \text{ is unramified.}$$

- H_0 is the classical Ramanujan–Selberg conjecture
- $H_{\frac{7}{64}}$ was proved by Kim–Sarnak–Shahidi (2003)

Theorem (BHM, 2004). Assume Hypothesis H_θ for any $0 \leqslant \theta < \frac{1}{2}$. Let g be a primitive (holomorphic or Maass) cusp form of level D and arbitrary nebentypus, and let χ be a primitive character of conductor q. For any $\varepsilon > 0$ and for $\Re s = \frac{1}{2}$ one has

$$L(s, g \otimes \chi) \ll_\varepsilon (|s|\mu_g D q)^\varepsilon |s|^A \mu_g^B D^C q^{\frac{1}{2}-\frac{1}{8}(1-2\theta)},$$

where

$$A := \frac{31+4\theta}{16}, \qquad B := \frac{73+12\theta}{16}, \qquad C := \frac{9}{16},$$

$$\mu_g := \begin{cases} 1 + \frac{k_g - 1}{2} & \text{if } g \text{ is a holomorphic form of weight } k_g, \\ 1 + |t_g| & \text{if } g \text{ is a Maass form of eigenvalue } \frac{1}{4} + t_g^2. \end{cases}$$

Approximate functional equation

$$L(s, g \otimes \chi) \quad \rightsquigarrow \quad N^{-1/2}\Sigma(N, g \otimes \chi), \quad N \ll_{s,g,\varepsilon} q^{1+\varepsilon},$$

where

$$\Sigma(N, g \otimes \chi) := \sum_{n \geqslant 1} \lambda_g(n)\chi(n)W_{N,s}(n)$$

for some smooth function $W_{N,s}$ supported in $[N, 2N]$ satisfying

$$x^i W_{N,s}^{(i)}(x) \ll_i |s|^i.$$

Amplification

For every character χ' modulo q define

$$\Sigma(N, g \otimes \chi') := \sum_{n \geqslant 1} \lambda_g(n) \chi'(n) W_{N,s}(n),$$

and consider the amplified second mean for $L \asymp q^\eta$ ($\eta > 0$ fixed):

$$\sum_{\chi' \,(\mathrm{mod}\, q)} \Bigg| \sum_{L \leqslant \ell \leqslant 2L} \overline{\chi}(\ell) \chi'(\ell) \Bigg|^2 \Big| \Sigma(N, g \otimes \chi') \Big|^2.$$

By orthogonality of characters, this is at most

$$\varphi(q) \sum_{L \leqslant \ell_1, \ell_2 \leqslant 2L} \overline{\chi}(\ell_1) \chi(\ell_2) \sum_h \sum_{\ell_1 m - \ell_2 n = hq} \overline{\lambda_g}(m) \lambda_g(n) \overline{W_{N,s}}(m) W_{N,s}(n).$$

Overview

$$\Sigma(N, g\otimes\chi) \ll_{g,\varepsilon} q^{1/2+\varepsilon}L^{-1/2}N^{1/2}$$
$$+\, q^{1/2+\varepsilon}\max_{\ell_1,\ell_2}\left|\sum_{h\neq 0}\ \sum_{\ell_1 m-\ell_2 n=hq}\overline{\lambda_g}(m)\lambda_g(n)\overline{W_{N,s}}(m)W_{N,s}(n)\right|^{1/2}$$

Blomer's optimal pointwise bound in $h \neq 0$ gives

$$\Sigma(N, g\otimes\chi) \ll_{g,\varepsilon} q^{1/2+\varepsilon}L^{-1/2}N^{1/2} + q^{2\varepsilon}(NL)^{(3+2\theta)/4}.$$

By averaging over $h \neq 0$ we improve this to

$$\Sigma(N, g\otimes\chi) \ll_{g,\varepsilon} q^{1/2+\varepsilon}L^{-1/2}N^{1/2} + q^{(1+2\theta)/4+2\varepsilon}(NL)^{1/2}.$$

Averages of shifted convolution sums

$$\mathcal{D}(g,\ell_1,\ell_2,q) := \sum_{h\neq 0}\phi(qh)\sum_{\ell_1 m-\ell_2 n=qh}\overline{\lambda_g}(m)\lambda_g(n)F(\ell_1 m,\ell_2 n)$$

$$= \int_0^1 H(\alpha)K(\alpha)\,d\alpha,$$

where

$$H(\alpha) := \sum_{h\neq 0}\phi(qh)e\Big(-\alpha qh)$$

$$K(\alpha) := \sum_{m,n\geqslant 1}\overline{\lambda_g}(m)\lambda_g(n)F(\ell_1 m,\ell_2 n)e\Big(\alpha(\ell_1 m-\ell_2 n)\Big)$$

Jutila's circle method

We look at intervals of fixed radius centered at rational points a/c with least denominators divisible by $D' := D\ell_1\ell_2$ and of prescribed size C. The contribution of such an interval has the form

$$\sum_{h\neq 0} e\left(\frac{-aqh}{c}\right) \sum_{m,n} \overline{\lambda_g}(m)\lambda_g(n)e\left(\frac{a\ell_1 m}{c}\right) e\left(\frac{-a\ell_2 n}{c}\right) E(m,n,h).$$

The original circle integral can be approximated by averages of such contributions. The error can be estimated by $\|H\|_2$, $\|K\|_\infty$, D' and C.

Voronoi summation

By applying Voronoi summation in the variables m, n and summing over a, c we reduce the estimation of $\mathcal{D}(g, \ell_1, \ell_2, q)$ to bounding certain averages of Kloosterman sums:

$$\sum_{c \equiv 0\,(D')} \sum_{h \neq 0} \sum_{h'} \frac{S(qh, h'; c)}{c} \sum_{\ell_1 m - \ell_2 n = h'} \overline{\lambda_g}(m) \lambda_g(n) \mathcal{E}(m, n, h; c).$$

Here $\mathcal{E}(m, n, h; c)$ is a Hankel-type transform of the previous weight function $E(x, y, h)$. The kernel involves Bessel functions depending on the cusp form g.

Sums of Kloosterman sums

The coefficients of the Kloosterman sums themselves are averages of shifted convolution sums

$$b_{y,h'} := \sum_{\substack{n \leqslant y \\ \ell_1 m - \ell_2 n = h'}} \overline{\lambda_g}(m)\lambda_g(n).$$

It suffices to estimate averages of Kloosterman sums of the form

$$\sum_{c \equiv 0\,(D')} \sum_{h>0} \sum_{h'>0} b_{y,h'} \frac{S(qh,h';c)}{c} g_y(qh,h';c).$$

If the original $F(x,y)$ is smooth and supported on $[X,2X] \times [Y,2Y]$, then (assuming $X \leqslant Y$) here the relevant range is

$$qh \ll Y, \quad h' \ll_{g,\varepsilon} P^{\varepsilon}\frac{C^2}{X}, \quad c \sim C, \quad y \ll_{g,\varepsilon} P^{\varepsilon}\frac{C^2}{\ell_2 Y}.$$

Kuznetsov's trace formula

For each h and h', the average of Kloosterman sums in the c variable can be expressed, by Kuznetsov's formula, as an average of products $\overline{\rho_f}(qh)\rho_f(h')$, where f runs through the entire spectrum of $\Gamma_0(D')\backslash \mathrm{SL}_2(\mathbb{R})$ and ρ_f denotes Fourier coefficient. By performing the averaging over h and h' we encounter sums of the form

$$\left(\sum_{h\sim H} \overline{\rho_f}(qh)\right) \times \left(\sum_{h'\sim H'} b_{y,h'}\rho_f(h')\right).$$

Spectral large sieve

We apply Cauchy–Schwartz and estimate the averages of

$$\left|\sum_{h\sim H}\overline{\rho_f}(qh)\right|^2 \quad \text{and} \quad \left|\sum_{h'\sim H'} b_{y,h'}\rho_f(h')\right|^2$$

separately using the spectral large sieve of Dehouillers and Iwaniec.

In the first average we need to extract, using the multiplicative properties of ρ_f, the factor q. This is by no means trivial, since q is in general not coprime with D' and Eisenstein series are in general not Hecke eigenforms.

Spectral large sieve

In the second average we make use of the strong square mean bound available for the shifted convolution sums

$$b_{y,h'} := \sum_{\substack{n \leqslant y \\ \ell_1 m - \ell_2 n = h'}} \overline{\lambda_g}(m) \lambda_g(n).$$

This bound is a consequence of the identity

$$\sum_{h' \in \mathbf{Z}} \left| \sum_{\substack{m \leqslant x,\ n \leqslant y \\ l_1 m - l_2 n = h'}} \overline{\lambda_g}(m) \lambda_g(n) \right|^2 = \int_0^1 \left| S_g(\ell_1 \alpha, x) S_g(\ell_2 \alpha, y) \right|^2 d\alpha,$$

where

$$S_g(\alpha, x) := \sum_{n \leqslant x} \lambda_g(n) e(n\alpha).$$