The Burgess bound for twisted Hilbert modular L-functions

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15 March 2010
Workshop on Analytic Number Theory
Institute for Advanced Study

Overview of talk

1. Subconvexity for twisted GL_2 L-functions (2 slides)
2. Applications
3. Illustration
4. Summary of new results
5. Main ingredients of the proof
6. Spectral decomposition of convolution sums (2 slides)
7. The proof in a nutshell (5 slides)

Subconvexity for twisted GL_2 L-functions over $\mathbb{Q}$

- s a point on the critical line ($\Re s = \frac{1}{2}$)
- f a primitive holomorphic or Maass cusp form
- χ a primitive Dirichlet character of conductor q

Lindelöf Hypothesis (follows from GRH)

For any $\delta < \frac{1}{2}$ we have $L(s, f \otimes \chi) \ll_{s,f,\delta} q^{\frac{1}{2}-\delta}$.

- $\delta < \frac{1}{22}$ (Duke–Friedlander–Iwaniec 1993, Michel 2004)
- $\delta < \frac{1}{54}$ (Harcos 2003)
- $\delta < \frac{1-2\theta}{10+4\theta}$ (Blomer 2004)
- $\delta < \frac{1-2\theta}{8}$ (Blomer–Harcos–Michel 2007)
- $\delta < \frac{1}{8}$ for f of trivial nebentypus (Bykovskii 1996, Blomer–Harcos 2008)
- $\delta < \frac{1}{6}$ for $s = \frac{1}{2}$, f self-dual, χ real (Conrey–Iwaniec 2000)

Subconvexity for twisted GL_2 L-functions over K

- K a totally real number field
- π an irreducible cuspidal automorphic representation of GL_2 over K with unitary central character
- χ a Hecke character of conductor $\mathfrak{q}$

Lindelöf Hypothesis (follows from GRH)

For any $\delta < \frac{1}{2}$ we have $L(\frac{1}{2}, \pi \otimes \chi) \ll_{K,\pi,\chi_\infty,\delta} (\mathcal{N}\mathfrak{q})^{\frac{1}{2}-\delta}$.

- $\delta < \frac{1-2\theta}{14+4\theta}$ for π induced by a totally holomorphic Hilbert modular form (Cogdell–Piatetski-Shapiro–Sarnak 2000)
- $\delta < \frac{(1-2\theta)^2}{14-12\theta}$ (Venkatesh 2005)
- $\delta < \frac{1-2\theta}{8}$ (Blomer–Harcos 2009)

Applications

- Number and distribution of representations by a totally positive integral ternary quadratic form
  1. Auxiliary results: Siegel 1935, Shimura 1973, Waldspurger 1981, Schulze-Pillot 1984, Duke–Schulze-Pillot 1990, Baruch–Mao 2007
  2. Core results: Siegel 1935, Linnik 1968, Iwaniec 1987, Duke 1988, Cogdell–Piatetski-Shapiro–Sarnak 2000
- Distribution of special subvarieties on Hilbert modular varieties (Linnik 1968, Duke 1988, Cohen 2005, Zhang 2005)
- First moment of central values of certain Hecke L-functions (Rodriguez-Villegas–Yang 1999, Kim–Masri–Yang 2010)
- Ingredient for GL_2 and $\mathrm{GL}_2 \times \mathrm{GL}_2$ subconvexity (Michel 2004, Harcos–Michel 2006, Blomer–Harcos–Michel 2007, Michel–Venkatesh 2010)

Illustration

- Interested in $r(n, Q)$ for $Q(x, y, z) := x^2 + y^2 + 10z^2$.
- Assume for simplicity that n is square-free and coprime to 10.
- Genus of Q contains another class represented by $Q'(x, y, z) := 2x^2 + 2y^2 + 3z^2 - 2xz$.
- $r(n, Q) + 2r(n, Q') = 2h(-10n) = n^{\frac{1}{2}+o(1)}$ by Siegel.
- Need to understand $r(n, Q) - r(n, Q')$.
- $(r(n, Q) - r(n, Q'))^2 = cn^{\frac{1}{2}} L(\frac{1}{2}, f \otimes (\frac{n}{\cdot}))$ for some constant $c > 0$ and some fixed primitive form $f \in S_2(\Gamma_0(1600))$.
- $L(\frac{1}{2}, f \otimes (\frac{n}{\cdot})) \ll n^{\frac{1}{2}-\delta}$ yields $r(n, Q) = \frac{2}{3}h(-10n) + O(n^{\frac{1-\delta}{2}})$.
- Under GRH the error term is $n^{\frac{1}{4}+o(1)}$ (good lower bounds by Hoffstein–Lockhart 1999, Rudnick–Soundararajan 2005).

Theorem (Ono–Soundararajan 1997)

Assume GRH. Assume n is not of the form $4^k(16m + 6)$ and not contained in the finite list 3, 7, 21, 31, 33, 43, 67, 79, 87, 133, 217, 219, 223, 253, 307 391, 679, 2719. *Then* $r(n, Q) > 0$.

Summary of new results

- K a totally real number field
- π an irreducible cuspidal automorphic representation of GL_2 over K with unitary central character
- χ a Hecke character of conductor $\mathfrak{q}$
- Q a totally positive integral ternary quadratic form over K

Theorem (Blomer–Harcos 2009, to appear in GAFA)

For any $\delta < \frac{1-2\theta}{8}$ *we have* $L(\frac{1}{2}, \pi \otimes \chi) \ll_{K,\pi,\chi_\infty,\delta} (\mathcal{N}\mathfrak{q})^{\frac{1}{2}-\delta}$.

Corollary

If n is a totally positive square-free integer in K which is integrally represented by Q over every completion of K, then

$$r(n, Q) = (\mathcal{N}n)^{\frac{1}{2}+o(1)} + O_{K,Q}((\mathcal{N}n)^{\frac{7}{16}+\frac{\theta}{8}+o(1)}),$$

where the main term is furnished by Siegel's mass formula.

Main ingredients of the proof

1. Approximate functional equation
2. Amplification method of Duke–Friedlander–Iwaniec
3. Spectral decomposition of shifted convolution sums
4. Contribution of continuous spectrum bounded using a good orthogonal basis of Eisenstein series
5. Contribution of discrete spectrum bounded using Venkatesh's variant of the Bruggeman–Kuznetsov formula

Spectral decomposition of convolution sums (1 of 2)

- π_1 and π_2 cuspidal representations of GL_2 over K whose central characters are unitary and inverse to each other
- $\phi_1 \in \pi_1$ and $\phi_2 \in \pi_2$ smooth cusp forms

Idea of linearization

Decompose the product $\phi_1\phi_2$ spectrally in $L^2(\mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbb{A}))$:

$$\phi_1\phi_2 = \int_{\varpi} \phi_{\varpi}\, d\varpi, \qquad \phi_{\varpi} \in \varpi.$$

Take Fourier-Whittaker coefficients on both sides. Left hand side becomes a convolution in the Hecke eigenvalues of π_1 and π_2. Right hand side becomes a combination of Hecke eigenvalues of the various ϖ's. Use the Kirillov model to generate any convolution sum. Use Sobolev norms and Plancherel to control the spectral coefficients in the decomposition.

Spectral decomposition of convolution sums (2 of 2)

Theorem (Jacobi 1829)

$$\sigma_k(n) := \sum_{d|n} d^k.$$

$$\sigma_3(q) + 120 \sum_{m+n=q} \sigma_3(m)\sigma_3(n) = \sigma_7(q).$$

Proof.

The spaces $M_4(\mathrm{SL}_2(\mathbb{Z}))$ and $M_8(\mathrm{SL}_2(\mathbb{Z}))$ are one-dimensional, generated by the Eisenstein series

$$E_4 = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) e(nz), \qquad E_8 = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) e(nz),$$

respectively. In particular, $E_4^2 = E_8$. The identity in the Theorem follows by taking q-th Fourier coefficients of both sides. □

The proof in a nutshell (1 of 5)

Combining the approximate functional equation with some ideas of Cogdell–Piatetski-Shapiro–Sarnak we reduce the Burgess bound to cancellation in certain finite sums:

$$\mathcal{L}_{\chi_{\mathrm{fin}}} \ll (\mathcal{N}\mathfrak{q})^{\frac{1}{2}-\frac{1}{8}(1-2\theta)+\varepsilon}.$$

Here we write, for any character $\xi : (\mathfrak{o}/\mathfrak{q})^\times \to S^1$,

$$\mathcal{L}_\xi := \sum_{0<<r\in\mathfrak{y}} \frac{\lambda_\pi(r\mathfrak{y}^{-1})\xi(r)}{\sqrt{\mathcal{N}(r\mathfrak{y}^{-1})}} W\left(\frac{r}{Y^{1/d}}\right),$$

where

- $\mathfrak{y}$ is an ideal class representative,
- $W : K_{\infty,+}^\times \to \mathbb{C}$ is some smooth function of compact support,
- $Y \ll (\mathcal{N}\mathfrak{q})^{1+\varepsilon}$.

By the amplification method of Duke–Friedlander–Iwaniec we see, for any amplifier length $L > 0$,

$$\frac{|\mathcal{L}_{\chi_{\mathrm{fin}}}|^2}{(\mathcal{N}\mathfrak{q})^{1+\varepsilon}} \ll \frac{1}{L} +$$

$$\sum_{0 \neq q \in \mathfrak{q}\mathfrak{y} \cap \mathcal{B}} \sum_{\substack{\ell_1 r_1 - \ell_2 r_2 = q \\ 0 \neq r_1, r_2 \in \mathfrak{y}}} \frac{\lambda_\pi(r_1 \mathfrak{y}^{-1}) \bar{\lambda}_\pi(r_2 \mathfrak{y}^{-1})}{\sqrt{\mathcal{N}(r_1 r_2 \mathfrak{y}^{-2})}} W\left(\frac{r_1}{Y^{1/d}}\right) \bar{W}\left(\frac{r_2}{Y^{1/d}}\right),$$

where

- $\mathcal{B} \subset K_\infty$ is some box of dimensions $\approx (LY)^{1/d}$,
- (ℓ_1) and (ℓ_2) are some prime ideals of norms $\approx L$.

We are dealing with a sum of shifted convolution sums. We decompose each of them spectrally:

$$\sum_{0 \neq q \in \mathfrak{q}\mathfrak{y} \cap \mathcal{B}} \int_{(\mathfrak{c})} \sum_{\mathfrak{t} | \mathfrak{c}\mathfrak{c}_{\varpi}^{-1}} \frac{\lambda_{\varpi}^{(\mathfrak{t})}(q\mathfrak{y}^{-1})}{\sqrt{\mathcal{N}(q\mathfrak{y}^{-1})}} W_{\varpi,\mathfrak{t}}\left(\frac{q}{(LY)^{1/d}}\right) d\varpi,$$

where the integral and sum are restricted to level

$$\mathfrak{c} := \mathfrak{c}_{\pi} \operatorname{lcm}((\ell_1),(\ell_2)).$$

Continuous spectrum contributes $\ll (\mathcal{N}\mathfrak{q})^{-1/2+\varepsilon} L^{1/2}$ by

$$\int_{\varpi \in \mathcal{E}(\mathfrak{c})} \sum_{\mathfrak{t} | \mathfrak{c}\mathfrak{c}_{\varpi}^{-1}} |W_{\varpi,\mathfrak{t}}(y)| \, d\varpi \ll (\mathcal{N}(\ell_1\ell_2))^{\varepsilon}$$

$$\lambda_{\varpi}^{(\mathfrak{t})}(q\mathfrak{y}^{-1}) \ll (\mathcal{N}\gcd(\mathfrak{t}, q\mathfrak{y}^{-1}))(\mathcal{N}(q\mathfrak{y}^{-1}))^{\varepsilon}$$

Most of the cuspidal contribution is negligible, thanks to

$$\int_{\varpi\in\mathcal{C}(\mathfrak{c})} (\mathcal{N}\tilde{\lambda}_\varpi)^A \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}} |W_{\varpi,\mathfrak{t}}(y)|\, d\varpi \ll_A |\mathcal{N}(\ell_1\ell_2)|^{\frac{1}{2}+\varepsilon}$$

We restrict to $\mathcal{N}\tilde{\lambda}_\varpi \leqslant (\mathcal{N}\mathfrak{q})^\varepsilon$ and separate variables in $W_{\varpi,\mathfrak{t}}$ by Mellin transforms. In this way we bound the cuspidal part by

$$(\mathcal{N}\mathfrak{q})^\varepsilon \left(\sum_{\substack{\varpi\in\mathcal{C}(\mathfrak{c},\varepsilon)\\ \mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}}} \left| \sum_{\mathcal{N}\mathfrak{m}\ll LY/\mathcal{N}(\mathfrak{q}\mathfrak{y})} \frac{\lambda_\varpi^{(\mathfrak{t})}(\mathfrak{m}\mathfrak{q})}{\sqrt{\mathcal{N}(\mathfrak{m}\mathfrak{q})}} f(\mathfrak{m}\mathfrak{q}) \right|^2 \right)^{1/2}$$

for some $f(\mathfrak{a}) \ll (\mathcal{N}\mathfrak{q})^\varepsilon$. Here we "almost factor out" $\lambda_\varpi^{(\mathfrak{t})}(\mathfrak{q})$ which is why we need θ: $|\lambda_\varpi(\mathfrak{q})| \ll (\mathcal{N}\mathfrak{q})^\theta$.

The proof in a nutshell (5 of 5)

The endgame:

1. Bound from above using smooth and rapidly decaying spectral weights.
2. Open the square and apply Venkatesh's variant of the Bruggeman-Kuznetsov formula.
3. Use familiar bounds of Weil for Kloosterman sums and Bruggeman–Miatello–Pacharoni for Bessel transforms.

Altogether amplification gives

$$\frac{|\mathcal{L}_{\chi_{\mathrm{fin}}}|^2}{(\mathcal{N}\mathfrak{q})^{1+\varepsilon}} \ll \frac{1}{L} + (\mathcal{N}\mathfrak{q})^{-1/2+\theta} L.$$

Right hand side is smallest when $L := (\mathcal{N}\mathfrak{q})^{\frac{1}{4}(1-2\theta)}$ in which case

$$\mathcal{L}_{\chi_{\mathrm{fin}}} \ll (\mathcal{N}\mathfrak{q})^{\frac{1}{2}-\frac{1}{8}(1-2\theta)+\varepsilon}.$$