F: number field of degree d

π: irreducible cuspidal automorphic representation of GL_m over F with unitary central character and contragradient representation $\tilde{\pi}$

$$\pi = \otimes_v \pi_v$$

$$\Lambda(s,\pi) = \prod_v L(s,\pi_v)$$

$$N^{\frac{s}{2}}\Lambda(s,\pi) = \kappa N^{\frac{1-s}{2}}\Lambda(1-s,\tilde{\pi})$$

N: conductor (a positive integer)

κ: root number (of modulus 1)

$$L(s,\pi_\infty) = \prod_{v|\infty} L(s,\pi_v) = \prod_{j=1}^{md} \pi^{\frac{\mu_j - s}{2}} \Gamma\left(\frac{s-\mu_j}{2}\right)$$

$$L(s,\pi_p) = \prod_{v|p} L(s,\pi_v) = \prod_{j=1}^{md} \frac{1}{1-\alpha_j(p)p^{-s}}$$

Theorem (Luo–Rudnick–Sarnak).

$$\sup\{\Re\mu_j, \Re \log_p \alpha_j(p)\} \leq \frac{1}{2} - \frac{1}{m^2+1}.$$

$$L(s,\pi) = \prod_{p<\infty} L(s,\pi_p) = \sum_{n=1}^{\infty} \frac{\lambda_\pi(n)}{n^s}, \quad \Re s > \frac{3}{2}.$$

$$C(s,\pi) = N \prod_{j=1}^{md} \frac{|s-\mu_j|}{2\pi}$$

Theorem (Molteni).

$$\sum_{n \le x} |\lambda_\pi(n)| \ll_{\epsilon,m,d} x^{1+\epsilon} C\left(\tfrac{1}{2}, \pi\right)^{\epsilon}.$$

Convexity Bound. *For any* $0 < \sigma < 1$,

$$L(\sigma + it, \pi) \ll_{\sigma,\epsilon,m,d} C\left(\tfrac{1}{2} + it, \pi\right)^{(1-\sigma)/2+\epsilon}.$$

Lindelöf Hypothesis. *For any* $0 < \sigma < 1$,

$$L(\sigma + it, \pi) \ll_{\sigma,\epsilon,m,d} C\left(\tfrac{1}{2} + it, \pi\right)^{\max(0,1-2\sigma)/2+\epsilon}.$$

$$S(X,\pi) = \sum_{n=1}^{\infty} \lambda_\pi(n) w\left(\frac{n}{X}\right)$$

$$S(X,\pi) \ll_{\epsilon,w,m,d} C\left(\tfrac{1}{2},\pi\right)^{1/4-\delta+\epsilon}\sqrt{X}$$

$$\frac{1}{|\mathcal{F}|}\sum_{\rho\in\mathcal{F}} |a_\rho|^2 |S(X,\rho)|^2 \ll_{\epsilon,w,m,d} C^\epsilon X$$

$$|\mathcal{F}| \ll C^{1/2+\epsilon}, \quad \pi\in\mathcal{F}, \quad |a_\pi| \gg C^\delta$$

$$D_f(a,b;h) = \sum_{am\pm bn=h} \lambda_\pi(m)\bar{\lambda}_\pi(n) f(am,bn)$$

Established cases of subconvexity over $\mathbb{Q}$

s: point on critical line ($\Re s = 1/2$)
χ: primitive Dirichlet character modulo q
f: holomorphic cusp form of full level
g: Maass cusp form of full level

$\zeta(s) \ll_\epsilon \lvert s\rvert^{1/4-1/12+\epsilon}$	**Weyl 1921**
$L(s,\chi) \ll_{\epsilon,s} q^{1/4-1/16+\epsilon}$	**Burgess 1963**
$L(s,\chi) \ll_\epsilon (\lvert s\rvert q)^{1/4-1/16+\epsilon}$	**HB 1980**
$L(s,f) \ll_{\epsilon,f} \lvert s\rvert^{1/2-1/6+\epsilon}$	**Good 1982**
$L(s,g) \ll_{\epsilon,g} \lvert s\rvert^{1/2-1/6+\epsilon}$	**Me 1990**
$L(s,f\otimes\chi) \ll_{\epsilon,s,f} q^{1/2-1/22+\epsilon}$	**Du–Fr–Iw 1993**

f: holomorphic cusp form of weight k, level N

$$L\left(\frac{1}{2}, f\right) \ll_{\epsilon,N} k^{1/2-1/6+\epsilon} \quad \textbf{Peng 2001}$$

f: Maass cusp form of eigenvalue $\frac{1}{4} + \mu^2$ and full level

$$L(s, f) \ll_{\epsilon,s} |\mu|^{1/2-1/6+\epsilon} \quad \textbf{Jutila 2002}$$

f: holomorphic or Maass newform of weight k, level N and trivial or primitive nebentypus

$$L(s, f) \ll_{\epsilon,s,k} N^{1/4-1/23040+\epsilon} \quad \textbf{Du–Fr–Iw 1994–2002}$$

f: holomorphic newform

g: holomorphic or Maass newform of level N and weight k

$L(s, f \otimes g) \ll_{\epsilon,s,f,N} k^{1-7/165+\epsilon}$ **Sarnak 2001**

$L(s, f \otimes g) \ll_{\epsilon,s,f,k} N^{1/2-1/1057+\epsilon}$ **Ko–Mi–Va 2001, Mi 2002**

f: Maass newform

g: holomorphic or Maass newform of level N and weight k

$L(s, f \otimes g) \ll_{\epsilon,s,f,N} k^{1-25/153+\epsilon}$ **Liu–Ye 2002**

$L(s, f \otimes g) \ll_{\epsilon,s,f,k} N^{1/2-?+\epsilon}$ **Ha–Mi (in progress)**

Theorem 1. *Suppose that ϕ is a primitive holomorphic or Maass cusp form of Archimedean size $|\tilde{\mu}|$, level N and arbitrary nebentypus character mod N. Let $\Re s = 1/2$ and q be an integer prime to N. If χ is a primitive Dirichlet character modulo q, then*

$$L(s, \phi \otimes \chi) \ll |s|^{1+\epsilon} N^{9/8+\epsilon} |\tilde{\mu}|^{27/20+\epsilon} q^{1/2-1/54+\epsilon},$$

where the implied constant depends only on ϵ.

Theorem 2. *Let $\lambda_\phi(m)$ (resp. $\lambda_\psi(n)$) be the normalized Fourier coefficients of a holomorphic or Maass cusp form ϕ (resp. ψ) of level N and arbitrary nebentypus character modulo N. Let $|\tilde{\mu}|$ (resp. $|\tilde{\nu}|$) denote the Archimedean size of ϕ (resp. ψ). Let $f(x,y)$ be a smooth weight function supported in $[A,2A]\times[B,2B]$ such that for some $P \geq 1$*

$$f^{(k,l)} \ll_{k,l} A^{-k}B^{-l}P^{k+l}, \quad k,l \geq 0.$$

Then for $h > 0$ and coprime $a > 0$ and $b > 0$ we have

$$\sum_{am-bn=h} \lambda_\phi(m)\lambda_\psi(n)f(am,bn) \ll$$

$$P^{11/10}N^{9/5}|\tilde{\mu}\tilde{\nu}|^{9/5+\epsilon}(ab)^{-1/10}(A+B)^{1/10}(AB)^{2/5+\epsilon},$$

where the implied constant depends only on ϵ.

$$\phi\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k\phi(z), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$$

$$\tilde{\mu} = \begin{cases} k/2 & \text{if } \phi \text{ is holomorphic} \\ 1/2 + i\mu & \text{if } \phi \text{ is real-analytic} \end{cases}$$

$$\phi(x+iy) = \sum_{n \neq 0} \rho_\phi(n) W(ny) e(nx)$$

$$W(y) = \begin{cases} e^{-2\pi y} & \text{if } \phi \text{ is holomorphic} \\ |y|^{1/2} K_{i\mu}\big(2\pi|y|\big) & \text{if } \phi \text{ is real-analytic} \end{cases}$$

$$\langle \phi, \phi \rangle = \int_{\Gamma_0(N)\backslash\mathcal{H}} y^k |\phi(x+iy)|^2 \, \frac{dx\,dy}{y^2}$$

$$\lambda_\phi(n) = \begin{cases} \left(\frac{N(k-1)!}{\langle \phi,\phi\rangle (4\pi n)^{k-1}} \right)^{1/2} \rho_\phi(n) & \text{if } \phi \text{ is holomorphic} \\ \left(\frac{N(4\pi|n|)}{\langle \phi,\phi\rangle \cosh \pi\mu} \right)^{1/2} \rho_\phi(n) & \text{if } \phi \text{ is real-analytic} \end{cases}$$

$$c_N \sum_{1 \le |n| \le x} |\lambda_\phi(n)|^2 \sim x \qquad \text{as } x \to \infty$$

$$c_N \asymp \frac{\mathrm{vol}\big(\Gamma_0(N)\backslash\mathcal{H}\big)}{N} = \frac{\pi}{3} \prod_{p|N} \left(1 + \frac{1}{p}\right)$$

$$\delta = P\frac{A+B}{AB}$$

$$F(x,y) = f(x,y)w(x-y-h)$$

$$|x-y-h| > \delta^{-1} \quad \Rightarrow \quad F(x,y) = 0$$

$$F^{(k,l)} \ll_{k,l} \delta^{k+l}, \quad k,l \geq 0$$

$$\sum_{am-bn=h} \lambda_\phi(m)\lambda_\psi(n)f(am,bn) = \int_0^1 G(\alpha)\,d\alpha$$

$$G(\alpha) = \sum_{m,n} \lambda_\phi(m)\lambda_\psi(n)F(am,bn)e\Big((am-bn-h)\alpha\Big)$$

Proposition 1 (Jutila). *Let $\mathcal{Q}$ be a nonempty set of integers $Q \le q \le 2Q$, where $Q \ge 1$. Let $Q^{-2} \le \delta \le Q^{-1}$, and for each fraction d/q (in its lowest terms) denote by $I_{d/q}(\alpha)$ the characteristic function of the interval $[d/q - \delta, d/q + \delta]$. Write L for the number of such intervals, that is,*

$$L = \sum_{q \in \mathcal{Q}} \varphi(q),$$

and put

$$\tilde{I}(\alpha) = \frac{1}{2\delta L} \sum_{q \in \mathcal{Q}} \ \sideset{}{^*}\sum_{d \pmod{q}} I_{d/q}(\alpha).$$

If $I(\alpha)$ is the characteristic function of the unit interval $[0, 1]$, then

$$\int_{-\infty}^{\infty} \Big(I(\alpha) - \tilde{I}(\alpha) \Big)^2 \, d\alpha \ll \delta^{-1} L^{-2} Q^{2+\epsilon},$$

where the implied constant depends on ϵ only.

$$\mathcal{Q} = \Big\{ q \in [Q, 2Q] : Nab \mid q \text{ and } (h,q) = (h, Nab) \Big\}$$

$$\sum_{am-bn=h} \lambda_\phi(m)\lambda_\psi(n) f(am, bn) = \frac{1}{2\delta L} \sum_{q\in\mathcal{Q}} \ \sideset{}{^*}\sum_{d \pmod q} \mathfrak{I}_{d/q} + \text{ error}$$

$$\mathfrak{I}_{d/q} = \int_{-\delta}^{\delta} G(d/q + \beta)\, d\beta$$

$$= e_q(-dh) \sum_{m,n} \lambda_\phi(m)\lambda_\psi(n) e_q\Big(d(am - bn)\Big) E(m, n)$$

$$E(x, y) = F(ax, by) \int_{-\delta}^{\delta} e\Big((ax - by - h)\beta\Big)\, d\beta$$

Lemma 1. *For any* $\epsilon > 0$ *there is a uniform bound*

$$\left\| y^{k/2}\phi(x+iy) \right\|_\infty \ll \langle \phi, \phi \rangle^{1/2} |\tilde{\mu}|^{3/2+\epsilon}.$$

The implied constant depends only on ϵ*.*

Proposition 2. *For any* $\epsilon > 0$ *there is a uniform bound*

$$\sum_{1 \le m \le M} \lambda_\phi(m) e(\alpha m) \ll N^{1/2} |\tilde{\mu}|^{2+\epsilon} M^{1/2+\epsilon}, \quad \alpha \in \mathbb{R},\ M > 0.$$

The implied constant depends only on ϵ*.*

Proposition 3 (Meurman). *Let d and q be coprime integers such that $N \mid q$, and let g be a smooth, compactly supported function on $(0, \infty)$. If ϕ is a real-analytic Maass cusp form of level N, nebentypus χ and nonnegative Laplacian eigenvalue $1/4 + \mu^2$ then*

$$\chi(d) \sum_{n=1}^{\infty} \lambda_\phi(n) e_q(dn) g(n) = \sum_{\pm} \sum_{n=1}^{\infty} \lambda_\phi(\mp n) e_q\left(\pm \bar{d} n\right) g^{\pm}(n),$$

where

$$g^{-}(y) = -\frac{\pi}{q \cosh \pi\mu} \int_0^\infty g(x) \{Y_{2i\mu} + Y_{-2i\mu}\} \left(\frac{4\pi\sqrt{xy}}{q}\right) dx,$$

$$g^{+}(y) = \frac{4 \cosh \pi\mu}{q} \int_0^\infty g(x) K_{2i\mu} \left(\frac{4\pi\sqrt{xy}}{q}\right) dx.$$

Here $\bar{d}$ is a multiplicative inverse of $d \bmod q$, $e_q(x) = e(x/q) = e^{2\pi i x/q}$, and $Y_{\pm 2i\mu}$, $K_{2i\mu}$ are Bessel functions.

$$\sum_{am-bn=h} \lambda_\phi(m)\lambda_\psi(n) f(am,bn) = \frac{1}{2\delta L} \sum_{q\in\mathcal{Q}} \sideset{}{^*}\sum_{d \pmod q} \mathfrak{I}_{d/q} + \text{ error}$$

$$\sideset{}{^*}\sum_{d \pmod q} \mathfrak{I}_{d/q} = \sum_{\pm\pm} \sum_{m,n\geq 1} \lambda_\phi(\mp m)\lambda_\psi(\mp n) S_{\overline{\chi\omega}}(-h, \pm am \mp bn; q) E^{\pm\pm}(m,n)$$

$$E^{\pm\pm}(m,n) = \frac{ab}{q^2} \int_0^\infty \int_0^\infty E(x,y) M^{\pm}_{2i\mu}\left(\frac{4\pi a\sqrt{mx}}{q}\right) M^{\pm}_{2i\nu}\left(\frac{4\pi b\sqrt{ny}}{q}\right) dx\, dy$$

$$M^{+}_{2ir} = (4\cosh \pi r) K_{2ir}, \qquad M^{-}_{2ir} = -\frac{\pi}{\cosh \pi r}\{Y_{2ir} + Y_{-2ir}\}.$$

We integrate by parts k times with respect to x, where $k = 0$ when m is small, and $k = \lceil 200/\epsilon \rceil$ when m is large. We integrate by parts l times with respect to y, where $l = 0$ when n is small, and $k = \lceil 200/\epsilon \rceil$ when n is large. Then we can achieve

$$E^{\pm\pm}(m,n) \ll_{k,l} \frac{|\tilde{\mu}\tilde{\nu}|^{\epsilon}(AB)^{1/2}}{\delta Q^2(A+B)} \left(\frac{A|\tilde{\mu}|^4(\delta Q)^2}{am}\right)^{\frac{k}{2}+\frac{1}{4}} \left(\frac{B|\tilde{\nu}|^4(\delta Q)^2}{bn}\right)^{\frac{l}{2}+\frac{1}{4}}$$

This shows that only small m and n contribute, and we conclude that

$$\sum_{d \pmod q}^{*} \mathfrak{I}_{d/q} = \sum_{\pm\pm} \sum_{m,n\geq 1} \lambda_\phi(\mp m)\lambda_\psi(\mp n) S_{\overline{\chi\omega}}(-h, \pm am \mp bn; q) E^{\pm\pm}(m,n)$$

is small.

Proposition 4. *For any $\sigma > 0$ and $\epsilon > 0$ the following uniform estimates hold in the strip $|\Re s| \leq \sigma$:*

$$e^{-\pi|\Im s|/2} Y_s(x) \ll \begin{cases} \Big(1+|\Im s|\Big)^{\sigma+\epsilon} x^{-\sigma-\epsilon}, & 0 < x \leq 1+|\Im s|; \\ \Big(1+|\Im s|\Big)^{-\epsilon} x^{\epsilon}, & 1+|\Im s| < x \leq 1+|s|^2; \\ x^{-1/2}, & 1+|s|^2 < x. \end{cases}$$

$$e^{\pi|\Im s|/2} K_s(x) \ll \begin{cases} \Big(1+|\Im s|\Big)^{\sigma+\epsilon} x^{-\sigma-\epsilon}, & 0 < x \leq 1+\pi|\Im s|/2; \\ e^{-x+\pi|\Im s|/2} x^{-1/2}, & 1+\pi|\Im s|/2 < x. \end{cases}$$

The implied constants depend only on σ and ϵ.

Altogether we get

$$\sum_{am-bn=h} \lambda_\phi(m)\lambda_\psi(n)f(am,bn) = \frac{1}{2\delta L}\sum_{q\in\mathcal{Q}} \ \sideset{}{^*}\sum_{d \ (\mathrm{mod}\ q)} \mathfrak{I}_{d/q} + \text{ error}$$

$$\ll \left\{ \frac{N^{3/2}|\tilde{\mu}\tilde{\nu}|^{3/2+\epsilon}\delta^2 Q^{3/2}}{ab} + \frac{N^2|\tilde{\mu}\tilde{\nu}|^{2+\epsilon}(ab)^{1/2}\delta^{1/2}}{Q} \right\} \frac{(AB)^{3/2+\epsilon}}{A+B}.$$

Optimal balance is achieved when $\delta^3 Q^5 \asymp N|\tilde{\mu}\tilde{\nu}|(ab)^3$, and this yields

$$\ll P^{11/10}N^{9/5}|\tilde{\mu}\tilde{\nu}|^{9/5+\epsilon}(ab)^{-1/10}(A+B)^{1/10}(AB)^{2/5+\epsilon}.$$