The density hypothesis for horizontal families of lattices

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3 July 2020
Number Theory during Lockdown

Collaborators and earlier talks

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- https://www.youtube.com/watch?v=65IIy_b6FkQ
- https://conferences.renyi.hu/ocaf2020/programme

Motivation and goal (1 of 2)

- G: Lie group $\mathrm{SL}_2(\mathbb{R})^a \times \mathrm{SL}_2(\mathbb{C})^b$
- Γ: arithmetic lattice in G
- $\widehat{G}_{\mathrm{sph}}$: spherical unitary dual of G
- $\mathrm{m}(\pi, \Gamma)$: multiplicity of $\pi \in \widehat{G}_{\mathrm{sph}}$ in $L^2(\Gamma \backslash G)$

Conjecture (Selberg)

If $\pi \in \widehat{G}_{\mathrm{sph}}$ is not tempered and not trivial, then $\mathrm{m}(\pi, \Gamma) = 0$.

Theorem (Sarnak–Xue 1991, Huntley–Katznelson 1993)

Let $a + b = 1$, and assume that $\Gamma(n)$ is a congruence subgroup of a fixed arithmetic lattice $\Gamma \leqslant G$. Let $\pi \in \widehat{G}_{\mathrm{sph}}$ have normalized Casimir eigenvalue $1/4 - \sigma^2$ with $\sigma \geqslant 0$. Then

$$\mathrm{m}(\pi, \Gamma(n)) \ll_{\varepsilon, \Gamma} \mathrm{vol}(\Gamma(n) \backslash G)^{1 - 2\sigma + \varepsilon}.$$

Motivation and goal (2 of 2)

- G: Lie group $\mathrm{SL}_2(\mathbb{R})^a \times \mathrm{SL}_2(\mathbb{C})^b$
- Γ: arithmetic lattice in G
- $\widehat{G}_{\mathrm{sph}}$: spherical unitary dual of G
- $\mathrm{m}(\pi, \Gamma)$: multiplicity of $\pi \in \widehat{G}_{\mathrm{sph}}$ in $L^2(\Gamma \backslash G)$

Goal

Prove a more general and more uniform version of the mentioned theorems of Sarnak–Xue (1991) & Huntley–Katznelson (1993):

1. Extend the multiplicity bound to G of rank $a + b \geqslant 2$.
2. Include more congruence subgroups of a fixed lattice Γ.
3. Get rid of the dependence on the fixed lattice Γ.
4. Address the dependence on the tempered components of π.
5. Talk about subsets of $\widehat{G}_{\mathrm{sph}}$, not just single representations π.

Congruence lattices (1 of 2)

- k: number field
- $\mathfrak{o}$: ring of integers of k
- $\mathfrak{p}$: nonzero prime ideal of $\mathfrak{o}$
- $\mathbb{A} = \mathbb{A}_\infty \times \mathbb{A}_f$: adele ring of k
- A: division quaternion algebra over k
- $\mathrm{ram}(A)$: finite set of places of k where A ramifies
- (a, b, c): determined by $A \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R})^a \times \mathrm{M}_2(\mathbb{C})^b \times \mathbb{H}^c$
- $D_{\mathfrak{p}}$: unique division quaternion algebra over $k_{\mathfrak{p}}$
- $\mathbf{G}$: algebraic group $\mathrm{SL}_1(A)$ defined over k

Fact

1. $\mathbf{G}(\mathbb{A}) \simeq \mathbf{G}(\mathbb{A}_\infty) \times \mathbf{G}(\mathbb{A}_f)$
2. $\mathbf{G}(\mathbb{A}_\infty) \simeq \mathbf{G}(k \otimes_{\mathbb{Q}} \mathbb{R}) \simeq \mathrm{SL}_2(\mathbb{R})^a \times \mathrm{SL}_2(\mathbb{C})^b \times \mathrm{SU}_2(\mathbb{C})^c$
3. $\mathbf{G}(\mathbb{A}_f) \simeq \prod_{\mathfrak{p} \in \mathrm{ram}(A)} \mathrm{SL}_1(D_{\mathfrak{p}}) \times \prod_{\mathfrak{p} \notin \mathrm{ram}(A)} \mathrm{SL}_2(k_{\mathfrak{p}})$

Congruence lattices (2 of 2)

- $\mathfrak{n}$: nonzero ideal of $\mathfrak{o}$ not divisible by any $\mathfrak{p} \in \mathrm{ram}(A)$
- $\mathfrak{n}_\mathfrak{p}$: closure of $\mathfrak{n}$ in $\mathfrak{o}_\mathfrak{p}$, that is, $\mathfrak{n}\mathfrak{o}_\mathfrak{p}$
- $K_0(\mathfrak{n}_\mathfrak{p})$: set of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathfrak{o}_\mathfrak{p})$ satisfying $c \in \mathfrak{n}_\mathfrak{p}$
- $K_1(\mathfrak{n}_\mathfrak{p})$: set of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathfrak{o}_\mathfrak{p})$ satisfying $a - d, b, c \in \mathfrak{n}_\mathfrak{p}$

Definition

1. For any map $\kappa\colon \{\mathfrak{p} \mid \mathfrak{n}\} \to \{0, 1\}$, let

$$K_\kappa(\mathfrak{n}) := \prod_{\mathfrak{p} \in \mathrm{ram}(A)} \mathrm{SL}_1(D_\mathfrak{p}) \times \prod_{\mathfrak{p} | \mathfrak{n}} K_{\kappa(\mathfrak{p})}(\mathfrak{n}_\mathfrak{p}) \times \prod_{\substack{\mathfrak{p} \notin \mathrm{ram}(A) \\ \mathfrak{p} \nmid \mathfrak{n}}} \mathrm{SL}_2(\mathfrak{o}_\mathfrak{p}).$$

   It is a compact open subgroup $\mathbf{G}(\mathbb{A}_f)$.

2. Let $\Gamma_\kappa(\mathfrak{n})$ be the intersection $\mathbf{G}(k) \cap (\mathbf{G}(\mathbb{A}_\infty) \times K_\kappa(\mathfrak{n}))$ projected to the factor $\mathrm{SL}_2(\mathbb{R})^a \times \mathrm{SL}_2(\mathbb{C})^b$ of $\mathbf{G}(\mathbb{A})$. It is an arithmetic lattice in G, and a congruence subgroup of $\Gamma_\emptyset(\mathfrak{o})$.

Main result

- j: element of $\{1, \ldots, a+b\}$ as an archimedean place of k
- ρ_j: degree of k_j over $\mathbb{R}$
- $\boldsymbol{s}$: $(a+b)$-tuple with components $s_j \in [-1/2, 1/2] \cup i\mathbb{R}$
- $\pi_{\boldsymbol{s}}$: element of $\widehat{G}_{\mathrm{sph}}$ induced from $\begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} \mapsto \prod_j |y_j|^{\rho_j s_j}$

Definition

Fix a partition $\{1, \ldots, a+b\} = S \cup S'$. Let $\boldsymbol{\sigma} = (\sigma_j) \in [0, 1/2]^S$ and $\boldsymbol{T} = (T_j) \in \mathbb{R}^{S'}$. Let $\Gamma \leqslant G$ be an arbitrary lattice.

1. Let $\mathcal{B}(\boldsymbol{\sigma}, \boldsymbol{T})$ be the set of $\pi_{\boldsymbol{s}} \in \widehat{G}_{\mathrm{sph}}$ such that $s_j \in [\sigma_j, 1/2]$ for all $j \in S$ and $s_j \in i[T_j - 1, T_j + 1]$ for all $j \in S'$.
2. Let $\mathcal{C}(\Gamma, \boldsymbol{T}) := \mathrm{vol}(\Gamma \backslash G) \prod_{j \in S'} (1 + |T_j|)^{\rho_j}$.

Theorem (Frączyk–Harcos–Maga–Milićević 2020)

$$\sum_{\pi \in \mathcal{B}(\boldsymbol{\sigma}, \boldsymbol{T})} \mathrm{m}(\pi, \Gamma_\kappa(\mathfrak{n})) \ll_{\varepsilon, a, b, c} \mathcal{C}(\Gamma_\kappa(\mathfrak{n}), \boldsymbol{T})^{\min_{j \in S}(1 - 2\sigma_j) + \varepsilon}$$

Overview of the proof (1 of 3)

1. We write $\mathbf{G}(\mathbb{A}_\infty) \times K_\kappa(\mathfrak{n})$ as $G \times U$, that is,
$$U := \mathrm{SU}_2(\mathbb{C})^c \times K_\kappa(\mathfrak{n}).$$
2. Using strong approximation, we identify the classical quotient $\Gamma_\kappa(\mathfrak{n})\backslash G$ with the adelic quotient $\mathbf{G}(k)\backslash\mathbf{G}(\mathbb{A})/U$.
3. We construct a positive definite test function $f \in C_c(G)$, with "controlled support and size", such that
$$\operatorname{tr}\pi(f) \gg \mathcal{C}(\Gamma_\kappa(\mathfrak{n}), \boldsymbol{T})^{2\max_{j\in S}\sigma_j}, \qquad \pi \in \mathcal{B}(\boldsymbol{\sigma}, \boldsymbol{T}).$$
4. We turn $f \in C_c(G)$ into $f_\mathbb{A} \in C_c(\mathbf{G}(\mathbb{A}))$ by tensoring it with $\mathbf{1}_U/\operatorname{vol}(U)$, where the Haar measure on $\mathrm{SU}_2(\mathbb{C})^c \times \mathbf{G}(A_f)$ is normalized so that maximal compact subgroups have mass 1.
5. It remains to bound, by $\mathcal{C}(\Gamma_\kappa(\mathfrak{n}), \boldsymbol{T})^{1+\varepsilon}$ from above, the trace
$$\begin{aligned}\operatorname{tr} R(f_\mathbb{A}) &= \sum_{\gamma\in\mathbf{G}(k)} \int_{\mathbf{G}(k)\backslash\mathbf{G}(\mathbb{A})} f_\mathbb{A}(g^{-1}\gamma g)\,dg\\ &= \sum_{[\gamma]\subset\mathbf{G}(k)} \operatorname{vol}(\mathbf{G}_\gamma(k)\backslash\mathbf{G}_\gamma(\mathbb{A})) \int_{\mathbf{G}_\gamma(\mathbb{A})\backslash\mathbf{G}(\mathbb{A})} f_\mathbb{A}(g^{-1}\gamma g)\,dg.\end{aligned}$$

Overview of the proof (2 of 3)

6. The contribution of the split conjugacy classes $[\pm \mathrm{id}]$ equals
$$\mathrm{vol}(\Gamma_\kappa(\mathfrak{n})\backslash G)(f(\mathrm{id}) + f(-\mathrm{id})) \ll \mathcal{C}(\Gamma_\kappa(\mathfrak{n}), \boldsymbol{T}).$$
7. We define $f \in C_c(G)$ as a pure tensor $\bigotimes_{j=1}^{a+b} f_j$, so $f_{\mathbb{A}} \in C_c(\mathbf{G}(\mathbb{A}))$ is a pure tensor $\bigotimes_v f_v$ with $f_v \in C_c(\mathbf{G}(k_v))$.
8. Hence the global orbital integral $\mathbf{O}(\gamma, f_{\mathbb{A}})$ decomposes as a product of local orbital integrals $\prod_v \mathbf{O}(\gamma, f_v)$, and we get
$$\mathbf{O}(\gamma, f_{\mathbb{A}}) \preccurlyeq |N_{k/\mathbb{Q}}(\Delta_{k(\gamma)/k})|^{-1/2} w_\kappa^{\mathfrak{n}}(\mathrm{tr}\,\gamma),$$
where $w_\kappa^{\mathfrak{n}} \colon \mathfrak{o} \to \mathbb{R}_{\geqslant 0}$ is a "mild function", and $\preccurlyeq$ means that we disregard factors of size $\mathcal{C}(\Gamma_\kappa(\mathfrak{n}), \boldsymbol{T})^{\varepsilon}$.
9. Combining the work of Ono (1961 & 1963) and Ullmo–Yafaev (2015) on algebraic tori, we also prove that
$$\mathrm{vol}(\mathbf{G}_\gamma(k)\backslash \mathbf{G}_\gamma(\mathbb{A})) \preccurlyeq \Delta_k^{1/2} |N_{k/\mathbb{Q}}(\Delta_{k(\gamma)/k})|^{1/2}.$$

Overview of the proof (3 of 3)

10. To summarize so far,

$$\operatorname{tr} R(f_{\mathbb{A}}) \preccurlyeq \mathcal{C}(\Gamma_\kappa(\mathfrak{n}), \boldsymbol{T}) + \Delta_k^{1/2} \sum_{[\gamma]\subset \mathbf{G}(k)}{}' \, w_\kappa^{\mathfrak{n}}(\operatorname{tr}\gamma),$$

    where the sum is over the regular semisimple conjugacy classes $[\gamma] \subset \mathbf{G}(k)$ satisfying $\mathbf{O}(\gamma, f_{\mathbb{A}}) \neq 0$.

11. Highly nontrivially, each trace $\operatorname{tr}(\gamma)$ occurs with multiplicity $\preccurlyeq 1$. Hence we are left with bounding the sum of $w_\kappa^{\mathfrak{n}}(x)$ over the possible traces $x \in \mathfrak{o}$ that occur. For this we invoke our earlier work in the geometry of numbers, and conclude that

$$\sum_{[\gamma]\subset \mathbf{G}(k)}{}' \, w_\kappa^{\mathfrak{n}}(\operatorname{tr}\gamma) \preccurlyeq \Delta_k^{-1/2} \mathcal{C}(\Gamma_\kappa(\mathfrak{n}), \boldsymbol{T}).$$

12. In the end, $\operatorname{tr} R(f_{\mathbb{A}}) \preccurlyeq \mathcal{C}(\Gamma_\kappa(\mathfrak{n}), \boldsymbol{T})$ follows, and we are done.

Some technical details

1. The construction of $f \in C_c(G)$ relies on the theory of the spherical transform on the groups $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$. We use that the spherical transform is the Mellin transform of the Harish-Chandra transform, and that orbital integrals can also be expressed in terms of the Harish-Chandra transform.

2. We estimate the non-archimedean local orbital integrals with the help of the Bruhat–Tits tree of $\mathrm{PGL}(k_{\mathfrak{p}})$. We are led to count various subgraphs of this tree, and for this we need to know explicitly the set of fixed points of an arbitrary regular semisimple element of $\mathrm{SL}_2(\mathfrak{o}_{\mathfrak{p}})$.

3. We use, e.g. in the geometry of numbers argument, that $\mathcal{C}(\Gamma_\kappa(\mathfrak{n}), \boldsymbol{T})$ is sufficiently large in terms of Δ_k and $N(\mathfrak{n})$. This information is provided by the volume formula of Borel (1981) that we adapt to our situation:

$$\mathrm{vol}(\Gamma_\emptyset(\mathfrak{o})\backslash G) = \frac{\zeta_k(2)\Delta_k^{3/2}}{2^{3b+2c}\pi^{a+2b+2c}} \prod_{\mathfrak{p}\in\mathrm{ram}(A)} (N(\mathfrak{p})-1).$$

Thanks for your attention!