Primes, Polignac, Polymath

Gergely Harcos

Alfréd Rényi Institute of Mathematics
http://www.renyi.hu/∼gharcos/

30 July 2019
Northeastern–BSM Colloquium

The ever sparser sequence of primes

10 digits	**100 digits**	**1000 digits**
1000000007	$100000\cdots000289$	$100000\cdots000007$
1000000009	$100000\cdots000303$	$100000\cdots000663$
1000000021	$100000\cdots000711$	$100000\cdots002121$
1000000033	$100000\cdots001287$	$100000\cdots002593$
1000000087	$100000\cdots002191$	$100000\cdots003561$
$\vdots$	$\vdots$	$\vdots$
9999999851	$999999\cdots997783$	$999999\cdots981127$
9999999881	$999999\cdots997873$	$999999\cdots988763$
9999999929	$999999\cdots998713$	$999999\cdots990139$
9999999943	$999999\cdots999089$	$999999\cdots993433$
9999999967	$999999\cdots999203$	$999999\cdots998231$
$\Delta \approx 22.3$	$\Delta \approx 229.5$	$\Delta \approx 2301.8$

The even sparser sequence of twin primes

10 digits	**100 digits**	**1000 digits**
1000000007	$1000\cdots00006001$	$1000\cdots01975081$
1000000009	$1000\cdots00006003$	$1000\cdots01975083$
1000000409	$1000\cdots00028441$	$1000\cdots03142729$
1000000411	$1000\cdots00028443$	$1000\cdots03142731$
$\vdots$	$\vdots$	$\vdots$
9999999017	$9999\cdots99914921$	$9999\cdots95309921$
9999999019	$9999\cdots99914923$	$9999\cdots95309923$
9999999701	$9999\cdots99964781$	$9999\cdots98131919$
9999999703	$9999\cdots99964783$	$9999\cdots98131921$

Twin prime conjecture

The equation $p - p' = 2$ has infinitely many solutions in primes.

Polignac numbers

Definition

A positive integer d is called a Polignac number if the equation $p - p' = d$ has infinitely many solutions in primes.

Conjecture (Polignac 1849)

Every positive even integer is a Polignac number.

Theorem (Zhang 2013)

One of $2, 4, 6, \ldots, 70000000$ *is a Polignac number.*

Theorem (Polymath 2014, Pintz 2013, Granville et al. 2014)

- *One of* $2, 4, 6, \ldots, 246$ *is a Polignac number.*
- *The lower density of Polignac numbers exceeds* $1/354$.
- *The gaps between Polignac numbers is bounded.*

Fishing for primes (1 of 2)

Idea

Let $\mathcal{H} = \{h_1, \ldots, h_k\}$ *be a* k*-set of integers. Try to find infinitely many positive integers* n *such that the translated set* $n + \mathcal{H} = \{n + h_1, \ldots, n + h_k\}$ *contains as many primes as possible.*

Definition

A k-set of integers is called admissible if it does not contain a complete system of residues modulo any integer bigger than one.

Idea

Let $\mathcal{H} = \{h_1, \ldots, h_k\}$ *be an admissible* k*-set. For any* $x > x_0$*, exhibit a probability measure on the integers* $x \leqslant n \leqslant 2x$ *such that the expected number of primes in* $n + \mathcal{H}$ *exceeds one.*
In other words, find nonnegative weights $\nu(n)$ *such that*

$$\sum_{x \leqslant n \leqslant 2x} \nu(n) \sum_{i=1}^{k} 1_{n + h_i \text{ is prime}} > \sum_{x \leqslant n \leqslant 2x} \nu(n).$$

Fishing for primes (2 of 2)

Conjecture (Dickson 1904, Hardy–Littlewood 1923)

Let $\mathcal{H}$ be an admissible k-set. Then for infinitely many positive integers n, the translated set $n + \mathcal{H}$ consists of k primes.

Theorem (Zhang 2013)

There exists a positive integer k with the following property. If $\mathcal{H}$ is an admissible k-set, then for infinitely many positive integers n, the translated set $n + \mathcal{H}$ contains at least two primes.

source	value of k	bound for prime gap
Zhang	3500000	70000000
Polymath8a	632	4680
Maynard	105	600
Polymath8b	50	246

The art of fishing (1 of 4)

The sifting weights of Goldston–Pintz–Yıldırım & Soundararajan

$$\nu(n) := \left(\sum_{d|(n+h_1)\dots(n+h_k)} \mu(d) g\left(\frac{\log d}{\log x^{\theta/2}}\right)\right)^2,$$

where $g : \mathbb{R} \to \mathbb{R}$ is sufficiently smooth and supported on $[0, 1]$. We restrict these weights to $x \leqslant n \leqslant 2x$ such that the prime factors of each $n + h_i$ exceed $\log\log\log x$.

Theorem (Goldston–Pintz–Yıldırım 2005, Soundararajan 2006)

Let $\mathcal{H}$ be an admissible k-set, and assume Hypothesis EH(θ). Then, for the probability measure determined by the above sifting weights, the expected number of primes in $n + \mathcal{H}$ equals

$$\frac{\theta}{2} \cdot \frac{k \int_0^1 g^{(k-1)}(t)^2 \ \frac{t^{k-2}}{(k-2)!} \ dt}{\int_0^1 g^{(k)}(t)^2 \ \frac{t^{k-1}}{(k-1)!} \ dt} + o(1).$$

Intermezzo: the Elliott–Halberstam conjecture

Hypothesis $EH(\theta)$

For any $A > 0$ there is a constant $C > 0$ such that, for any $x \geqslant 2$,

$$\sum_{\substack{q \leqslant x^\theta \\ q \text{ squarefree}}} \max_{(a,q)=1} \left| \sum_{\substack{x \leqslant p \leqslant 2x \\ p \equiv a \pmod q}} 1 - \frac{1}{\varphi(q)} \int_x^{2x} \frac{dt}{\log t} \right| < C \frac{x}{\log^A x}.$$

Remarks

- *True for $\theta < 1/2$ by Bombieri (1965) & Vinogradov (1966).*
- *Conjectured for $\theta < 1$ by Elliott–Halberstam (1970).*

The art of fishing (2 of 4)

- Zhang established a weaker version of $EH(\theta)$ for any $\theta < 1/2 + 1/584$, by deep exponential sum methods. This allowed him to take $k = 3500000$, with a lot to spare.

- In the weaker version of $EH(\theta)$, both q and the residue class a modulo q are strongly restricted. For example, q is allowed to have small prime factors only. This idea goes back to Motohashi–Pintz (2008).

- The Polymath8a research group, led by Tao, relaxed the restriction on q and decreased its negative effect on k. Moreover, the exponential sum estimates of Zhang have been improved significantly. In the end, we could take any $\theta < 1/2 + 7/300$, leading to the value $k = 632$.

The art of fishing (3 of 4)

The sifting weights of Maynard & Tao

$$\nu(n) := \left(\sum_{\forall i: d_i | n+h_i} \mu(d_1) \dots \mu(d_k) f\left(\frac{\log d_1}{\log x^{\theta/2}}, \dots, \frac{\log d_k}{\log x^{\theta/2}} \right) \right)^2,$$

where $f : \mathbb{R}^k \to \mathbb{R}$ is a symmetric and sufficiently smooth function supported on the simplex $\{(t_1, \dots, t_k) \in \mathbb{R}^k_{\geqslant 0} : t_1 + \dots + t_k \leqslant 1\}$.

Theorem (Maynard 2013, Tao 2013)

Let $\mathcal{H}$ be an admissible k-set, and assume Hypothesis EH(θ). Then, for the probability measure determined by the above sifting weights, the expected number of primes in $n + \mathcal{H}$ equals

$$\frac{\theta}{2} \cdot \frac{k \int_{\mathbb{R}^{k-1} \times \{0\}} \left(\frac{\partial^{k-1} f}{\partial t_1 \dots \partial t_{k-1}} \right)^2}{\int_{\mathbb{R}^k} \left(\frac{\partial^k f}{\partial t_1 \dots \partial t_k} \right)^2} + o(1).$$

The art of fishing (4 of 4)

- For $f(t_1, \ldots, t_k) := g(t_1 + \cdots + t_k)$ the sifting weights of Maynard & Tao reduce to the sifting weights of Goldston–Pintz–Yıldırım & Soundararajan.
- The optimal $f : \mathbb{R}^k \to \mathbb{R}$ catches about $(\log k)/4$ primes.
- Further improvements are possible by enlarging the support of f or by incorporating the ideas of Zhang/Polymath8a.
- Symmetric polynomials f found by Maynard/Polymath8b with the help of computers show that Zhang's theorem holds for rather small k, the current record being $k = 50$.
- Under a suitably generalized Elliott–Halberstam conjecture Polymath8b could take $k = 3$, improving on the earlier values of $k = 5$ by Maynard and $k = 6$ by Goldston–Pintz–Yıldırım.