The sequence of prime gaps is graphic

(joint with P. L. Erdős, S. R. Kharel, P. Maga, T. R. Mezei, Z. Toroczkai)

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Introducing prime gap graphs

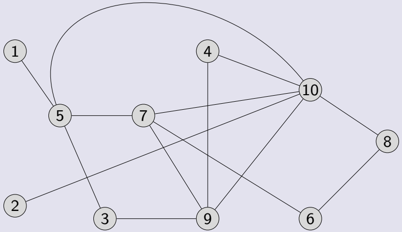
Definition

Let p_n denote the n-th prime number, and let $p_0 = 1$.

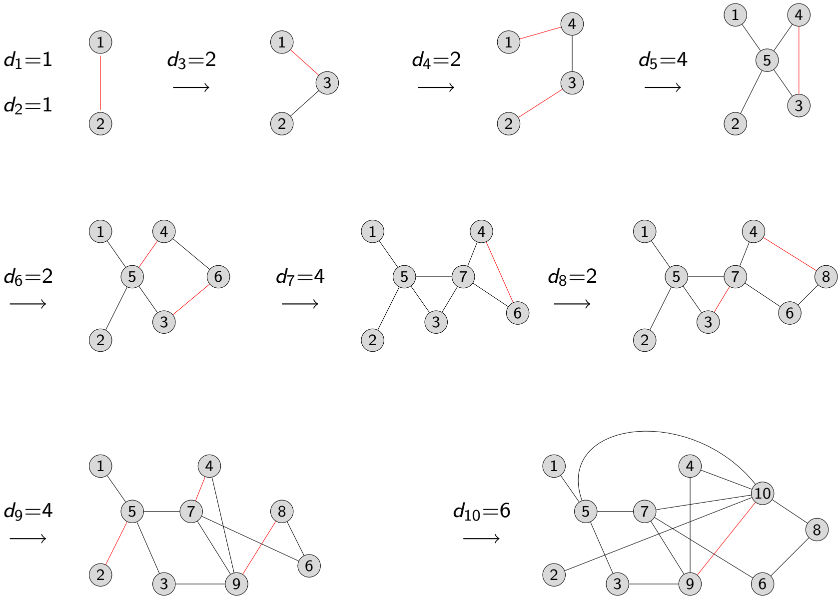
We call a simple graph on $n \geqslant 2$ vertices a *prime gap graph* if its vertex degrees are $p_1 - p_0, \ldots, p_n - p_{n-1}$.

Example ($n = 10$)

1	2	3	5	7	11	13	17	19	23	29
	1	1	2	2	4	2	4	2	4	6



Prime gap graphs generated by a DPG-process



Imagine that we made to 30 vertices...

- A gap of 14 occurs between $p_{30} = 113$ and $p_{31} = 127$. The earlier gaps are smaller, in fact they do not exceed 8.
- Imagine that we made to 30 vertices with the DPG-process. Then our prime gap graph has $(p_{30} - 1)/2 = 56$ edges.
- To continue, we want to remove $14/2 = 7$ independent edges, and connect their 14 ends to a new vertex, creating a prime gap graph with 31 vertices and $(p_{31} - 1)/2 = 63$ edges.
- How can we guarantee $14/2 = 7$ independent edges without actually looking at the graph?

Theorem (Vizing 1964)

The edges of a simple graph with maximal vertex degree Δ can be colored with $\Delta + 1$ colors.

Corollary

The 56 edges of a prime gap graph on 30 vertices can be colored with 9 colors. The largest color class has at least 7 members, because $9 \cdot 6 < 56$, and it consists of independent edges.

Main conjectures and main theorem

Conjecture (Toroczkai 2016)

For every $n \geqslant 2$, there exists a prime gap graph on n vertices.

Conjecture (Toroczkai 2016)

In every prime gap graph on n vertices, there exist $(p_{n+1} - p_n)/2$ independent edges.

Remark

The second conjecture says that, starting with the prime gap graph on 2 vertices, the DPG-process runs indefinitely. Hence it implies the first conjecture.

Theorem (EHKMMT 2022)

The above conjectures are true for every sufficiently large n. Assuming the Riemann hypothesis, they are true for every $n \geqslant 2$.

Existence of prime gap graphs under RH

Notation

We shall denote by G_n any prime gap graph on $n \geqslant 2$ vertices. It has $(p_n - 1)/2$ edges.

Theorem (EHKMMT 2022)

Assume the Riemann hypothesis. In every prime gap graph G_n on n vertices, there exist $(p_{n+1} - p_n)/2$ independent edges. Hence the DPG-process creates an infinite sequence $(G_2, G_3, \ldots,)$ of prime gap graphs.

Skeleton of the proof

Let N be a parameter. Delete all vertices of degree at least N (and the incident edges) from G_n. The remaining graph H_n admits an edge coloring with N colors, by the theorem of Vizing (1964). For suitable N, the largest color class has size at least $(p_{n+1} - p_n)/2$.

$(p_{n+1} - p_n)/2$ independent edges for $p_n < 10^{18}$

We can assume $n \geqslant 5$. For $p_n < 10^{18}$, we choose

$$N := \max_{1 \leqslant \ell \leqslant n} (1 + p_\ell - p_{\ell-1}).$$

That is, we apply Vizing's theorem to $H_n = G_n$. It suffices that

$$\left\lceil \frac{p_n - 1}{2N} \right\rceil \geqslant \frac{p_{n+1} - p_n}{2}.$$

For $n \leqslant 44$, this can be checked by a simple computer program. For $n \geqslant 45$, the statement is a consequence of the following

Lemma (cf. T. Oliveira e Silva, S. Herzog, S. Pardi 2014)

For any $x \in [117, 10^{18}]$, there is a prime number in $[x, x + \sqrt{x}]$.

Indeed, let $k \geqslant 15$ be the integer satisfying $(k-1)^2 < p_n < k^2$. The lemma implies that $p_{n+1} - p_n \leqslant k - 1$ and $N \leqslant k$, hence

$$\left\lceil \frac{p_n - 1}{2N} \right\rceil \geqslant \frac{k-1}{2} \geqslant \frac{p_{n+1} - p_n}{2}.$$

$(p_{n+1} - p_n)/2$ independent edges for $p_n > 10^{18}$

For $p_n > 10^{18}$, we choose

$$N := \left\lceil \frac{\sqrt{p_n}}{3 \log p_n} \right\rceil .$$

So we delete at most

$$\sum_{\substack{\ell \leqslant n \\ p_\ell - p_{\ell-1} \geqslant N}} (p_\ell - p_{\ell-1})$$

edges from G_n, and we apply Vizing's theorem to the remaining graph H_n. We shall see that the sum above is less than $(p_n - 1)/3$, hence H_n has more than $(p_n - 1)/6$ edges. Now it suffices to invoke

Theorem (Carneiro–Milinovich–Soundararajan 2019)

Assume the Riemann hypothesis. Then, for any $x \geqslant 4$, there is a prime number in $[x, x + \frac{22}{25}\sqrt{x}\log x]$.

Indeed, these results imply that

$$\left\lceil \frac{p_n - 1}{6N} \right\rceil > 0.499\sqrt{p_n}\log p_n > \frac{p_{n+1} - p_n}{2}.$$

Main analytic input under RH (1 of 3)

The claimed lower bound $(p_n - 1)/6$ for the number of edges of H_n follows from an explicit version of a result by Selberg (1943):

Theorem (EHKMMT 2022)

Assume the Riemann hypothesis. Then, for any $x \geqslant 2$ and $N > 0$, we have

$$\sum_{\substack{x \leqslant p_\ell \leqslant 2x \\ p_{\ell+1} - p_\ell \geqslant N}} (p_{\ell+1} - p_\ell) < \frac{163 x \log^2 x}{N}.$$

Indeed, for

$$p_n > 10^{18} \qquad \text{and} \qquad N := \left\lceil \frac{\sqrt{p_n}}{3 \log p_n} \right\rceil,$$

this theorem readily gives that

$$\sum_{\substack{\ell \leqslant n \\ p_\ell - p_{\ell-1} \geqslant N}} (p_\ell - p_{\ell-1}) < 489 \sqrt{p_n} \log^3 p_n < \frac{p_n - 1}{3}.$$

Main analytic input under RH (2 of 3)

The proof relies on ideas of Heath-Brown (1978) and Saffari–Vaughan (1977). First, one can restrict to $x > 10^{18}$ and

$$81 \log^2 x < N < \frac{4}{3}\sqrt{x}\log x.$$

Then, writing $N = 4\delta x$, the statement can be reduced to

$$\int_x^{2x} |\psi(y+\delta y) - \psi(y) - \delta y|^2 \, dy < 20\, \delta x^2 \log^2 x.$$

Now we employ an explicit version of a result by Goldston (1983):

Theorem (EHKMMT 2022)

For any $z > x > 10^{18}$ we have

$$\psi(x) = x - \sum_{|\Im\rho|<z} \frac{x^\rho}{\rho} + O^*(5\log x \log\log x),$$

where the sum is over the nontrivial zeros of the Riemann zeta function (counted with multiplicity).

Main analytic input under RH (3 of 3)

Then it remains to show that

$$\int_x^{2x} \left| \sum_{|\Im\rho|<3x} y^\rho C(\rho) \right|^2 dy < 9.942\delta x^2 \log^2 x,$$

where

$$C(\rho) := \frac{1-(1+\delta)^\rho}{\rho}.$$

Here the calculation becomes technical. In big steps:

$$\begin{aligned} \text{LHS} &< \int_1^2 \int_{xv/2}^{2xv} \left| \sum_{|\Im\rho|<3x} y^\rho C(\rho) \right|^2 dy\, dv \\ &< x^2 \sum_{\rho,\rho'} |C(\rho)|^2 \left| \frac{2^2+2^{-2}}{2+\rho-\rho'} \right| \left| \frac{2^3+1}{3+\rho-\rho'} \right| \\ &< 15.616 x^2 \sum_{\Im\rho>0} \min\left(\delta^2, \frac{4}{(\Im\rho)^2}\right) \left(\frac{1}{2} + \log\frac{\Im\rho}{2\pi}\right). \end{aligned}$$

Graphicality of the prime gap sequence without RH (1 of 5)

Theorem (EHKMMT 2022)

Let $n \geqslant 2$ be sufficiently large. There exists a prime gap graph on n vertices. Moreover, in every prime gap graph on n vertices, there exist $(p_{n+1} - p_n)/2$ independent edges. Hence the DPG-process creates an infinite sequence $(G_m, G_{m+1}, \ldots,)$ of prime gap graphs.

We deduce the first part from the following classical result.

Theorem (Erdős–Gallai 1960)

Let $d_1 \geqslant \cdots \geqslant d_n \geqslant 0$ be integers. Then the sequence $(d_1, \ldots, d_n)$ is graphic if and only if $d_1 + \cdots + d_n$ is even and for every $k \in \{1, \ldots, n\}$ we have

$$\sum_{\ell=1}^{k} d_\ell \leqslant k(k-1) + \sum_{\ell=k+1}^{n} \min(k, d_\ell) \, .$$

Interestingly, we can apply this result to a long initial segment of the prime gap sequence even though this sequence is not ordered.

Graphicality of the prime gap sequence without RH (2 of 5)

Theorem (EHKMMT 2022)

Let $\mathbf{D} = (d_1, \dots, d_n)$ *be a sequence of positive integers such that* $\|\mathbf{D}\|_1 = \sum_{\ell=1}^{n} d_\ell$ *is even. Let* $1 < p \leqslant \infty$ *be a parameter, and assume that the following* L^p*-norm bound holds:*

$$\|2 + \mathbf{D}\|_p \leqslant n^{\frac{1}{2} + \frac{1}{2p}}.$$

Then there is a simple graph G *with degree sequence* $\mathbf{D}$*.*

Proof (sketch).

By symmetry, we can assume that $d_1 \geqslant \cdots \geqslant d_n$. Denoting $\mathbf{D}^k := (d_1, \dots, d_k)$, we strengthen the Erdős–Gallai condition to $\|2 + \mathbf{D}^k\|_1 \leqslant k^2 + n$. This stronger condition follows from the initial assumption and Hölder's inequality, hence we are done:

$$\|2 + \mathbf{D}^k\|_1 \leqslant k^{1-\frac{1}{p}} \|2 + \mathbf{D}^k\|_p \leqslant k^{1-\frac{1}{p}} n^{\frac{1}{2} + \frac{1}{2p}} < k^2 + n. \qquad \square$$

Graphicality of the prime gap sequence without RH (3 of 5)

Applying the previous theorem with $p = 2$, it remains to verify that

$$\sum_{\ell=1}^{n} (2 + p_\ell - p_{\ell-1})^2 \leqslant n^{3/2}.$$

By Heath-Brown (1978), the left-hand side is at most $n^{4/3+o(1)}$, hence we are done.

We deduce the existence of $(p_{n+1} - p_n)/2$ independent edges in G_n from the theorem of Vizing (1964). In general, we have

Theorem (EHKMMT 2022)

Let $\mathbf{D} = (d_1, \ldots, d_n)$ be a sequence of positive integers such that $\|\mathbf{D}\|_1 = \sum_{\ell=1}^{n} d_\ell$ is even. Let $1 < p \leqslant \infty$ be a parameter, and let G be any simple graph with degree sequence $\mathbf{D}$. Assume that $d \geqslant 2$ is an even integer satisfying

$$4d^{1-\frac{1}{p}} \|\mathbf{D}\|_p \leqslant \|\mathbf{D}\|_1.$$

Then G contains $d/2$ independent edges.

Proof (sketch).

By Vizing's theorem, it suffices to verify that the following condition holds for some integer $\delta \geqslant 1$:

$$\frac{1}{\delta}\left(\frac{1}{2}\sum_{\ell=1}^{n} d_\ell - \sum_{d_\ell \geqslant \delta} d_\ell\right) \geqslant \frac{d}{2}.$$

If $p = \infty$, then we can choose $\delta := 1 + \|\mathbf{D}\|_\infty$. So let us focus on the case $1 < p < \infty$. For any integer $\delta \geqslant 1$, we have

$$\sum_{\ell=1}^{n} d_\ell - 2\sum_{d_\ell \geqslant \delta} d_\ell \geqslant \|\mathbf{D}\|_1 - 2\delta^{1-p}\|\mathbf{D}\|_p^p,$$

hence it suffices that

$$\delta^{1-p}\|\mathbf{D}\|_p^p \leqslant \frac{1}{4}\|\mathbf{D}\|_1 \qquad \text{and} \qquad \delta d \leqslant \frac{1}{2}\|\mathbf{D}\|_1.$$

Graphicality of the prime gap sequence without RH (5 of 5)

Proof (sketch, continued).

In other words, it suffices to find an integer δ satisfying

$$\left(\frac{4\|\mathbf{D}\|_p^p}{\|\mathbf{D}\|_1}\right)^{\frac{1}{p-1}} \leqslant \delta \leqslant \frac{1}{2d}\|\mathbf{D}\|_1.$$

The left-hand side exceeds 1, hence δ exists as long as

$$2\left(\frac{4\|\mathbf{D}\|_p^p}{\|\mathbf{D}\|_1}\right)^{\frac{1}{p-1}} \leqslant \frac{1}{2d}\|\mathbf{D}\|_1.$$

□

Applying the previous theorem with $p = 2$, it remains to verify that

$$16(p_{n+1} - p_n)\sum_{\ell=1}^{n}(p_\ell - p_{\ell-1})^2 \leqslant (p_n - 1)^2.$$

By Ingham (1937) and Heath-Brown (1978), the left-hand side is at most $n^{5/8+4/3+o(1)} = n^{47/24+o(1)}$, hence we are done.