The sequence of prime gaps is graphic

(joint with P. L. Erdős, S. R. Kharel, P. Maga, T. R. Mezei, Z. Toroczkai)

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Introducing prime gap graphs

Definition

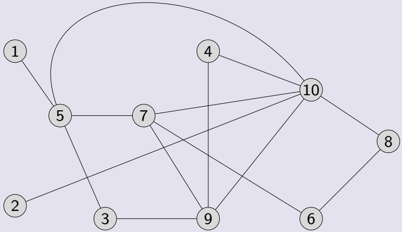
Let p_n denote the n-th prime number, and let $p_0 = 1$.

We call a simple graph on $n \geqslant 2$ vertices a *prime gap graph* if its vertex degrees are $p_1 - p_0, \ldots, p_n - p_{n-1}$.

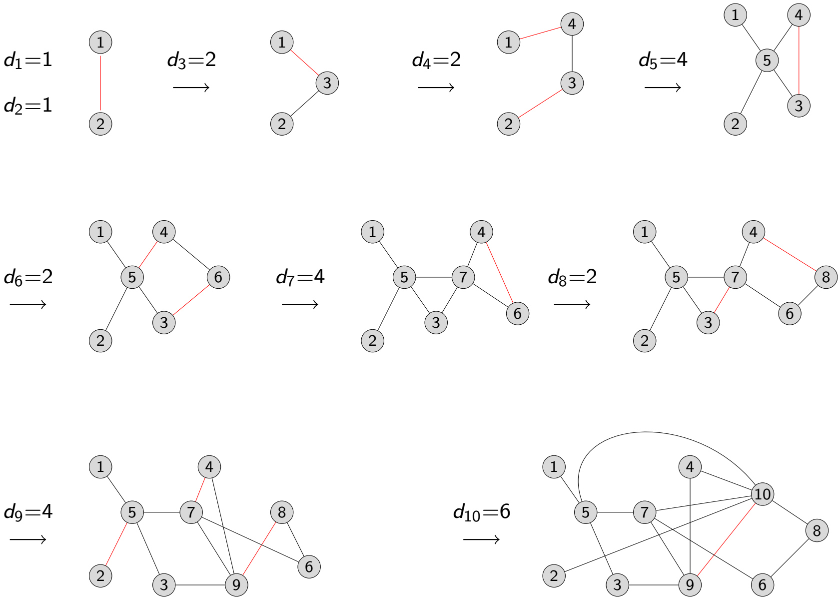
Example ($n = 10$)

1 2 3 5 7 11 13 17 19 23 29

1 1 2 2 4 2 4 2 4 6



Prime gap graphs generated by a DPG-process



Imagine that we made to 30 vertices...

- A gap of 14 occurs between $p_{30} = 113$ and $p_{31} = 127$. The earlier gaps are smaller, in fact they do not exceed 8.
- Imagine that we made to 30 vertices with the DPG-process. Then our prime gap graph has $(p_{30} - 1)/2 = 56$ edges.
- To continue, we want to remove $14/2 = 7$ independent edges, and connect their 14 ends to a new vertex, creating a prime gap graph with 31 vertices and $(p_{31} - 1)/2 = 63$ edges.
- How can we guarantee $14/2 = 7$ independent edges without actually looking at the graph?

Theorem (Vizing 1964)

The edges of a simple graph with maximal vertex degree Δ can be colored with $\Delta + 1$ colors.

Corollary

The 56 edges of a prime gap graph on 30 vertices can be colored with 9 colors. The largest color class has at least 7 members, because $9 \cdot 6 < 56$, and it consists of independent edges.

Main conjectures and main theorem

Conjecture (Toroczkai 2016)

For every $n \geqslant 2$, there exists a prime gap graph on n vertices.

Conjecture (Toroczkai 2016)

In every prime gap graph on n vertices, there exist $(p_{n+1} - p_n)/2$ independent edges.

Remark

The second conjecture says that, starting with the prime gap graph on 2 vertices, the DPG-process runs indefinitely. Hence it implies the first conjecture.

Theorem

The above conjectures are true for every sufficiently large n. Assuming the Riemann hypothesis, they are true for every $n \geqslant 2$.

$(p_{n+1} - p_n)/2$ independent edges under RH (1 of 2)

We can assume $n \geqslant 5$. By Vizing's theorem, it suffices that

$$\left\lceil \frac{p_n - 1}{2N} \right\rceil \geqslant \frac{p_{n+1} - p_n}{2},$$

where

$$N := \max_{1 \leqslant \ell \leqslant n} (1 + p_\ell - p_{\ell-1}).$$

For $5 \leqslant n \leqslant 44$ we checked this by a simple computer program. For $n \geqslant 45$, it would suffice to prove the following

Conjecture (cf. Oppermann 1877 & Legendre 1797)

For any $x \geqslant 117$, there is a prime number in $[x, x + \sqrt{x}]$.

Indeed, let $k \geqslant 15$ be the integer satisfying $(k-1)^2 < p_n < k^2$. The conjecture implies that $p_{n+1} - p_n \leqslant k - 1$ and $N \leqslant k$, hence

$$\left\lceil \frac{p_n - 1}{2N} \right\rceil \geqslant \frac{k-1}{2} \geqslant \frac{p_{n+1} - p_n}{2}.$$

$(p_{n+1} - p_n)/2$ independent edges under RH (2 of 2)

The previous conjecture holds for $x \in [117, 10^{18}]$ by known prime gap records, hence we can assume that $p_n > 10^{18}$. We shall use:

Theorem (Carneiro–Milinovich–Soundararajan 2019)

Assume the Riemann hypothesis. Then, for any $x \geqslant 4$, there is a prime number in $[x, x + \frac{22}{25}\sqrt{x}\log x]$.

Theorem (EHKMMT 2022)

Assume the Riemann hypothesis. Let $p_n > 10^{18}$, and let G be a prime gap graph on n vertices. For at least one-third of the edges of G, the endpoints have degrees less than

$$N := \left\lceil \frac{\sqrt{p_n}}{3\log p_n} \right\rceil .$$

We succeed by applying Vizing's theorem to a subgraph of G:

$$\left\lceil \frac{p_n - 1}{6N} \right\rceil > 0.499\sqrt{p_n}\log p_n > \frac{p_{n+1} - p_n}{2}.$$

Main analytic input under RH (1 of 3)

The last theorem follows from an explicit version of a result by Selberg (1943):

Theorem (EHKMMT 2022)

Assume the Riemann hypothesis. Then, for any $x \geqslant 2$ and $N > 0$, we have

$$\sum_{\substack{x \leqslant p_\ell \leqslant 2x \\ p_{\ell+1} - p_\ell \geqslant N}} (p_{\ell+1} - p_\ell) < \frac{163 x \log^2 x}{N}.$$

Indeed, for

$$p_n > 10^{18} \qquad \text{and} \qquad N := \left\lceil \frac{\sqrt{p_n}}{3 \log p_n} \right\rceil$$

we obtain

$$\sum_{\substack{\ell \leqslant n \\ p_\ell - p_{\ell-1} \geqslant N}} (p_\ell - p_{\ell-1}) < 489 \sqrt{p_n} \log^3 p_n < \frac{p_n - 1}{3}.$$

Main analytic input under RH (2 of 3)

The proof relies on ideas of Heath-Brown (1978) and Saffari–Vaughan (1977). First, one can restrict to $x > 10^{18}$ and

$$81 \log^2 x < N < \frac{4}{3}\sqrt{x}\log x.$$

Then, writing $N = 4\delta x$, the statement can be reduced to

$$\int_x^{2x} |\psi(y+\delta y) - \psi(y) - \delta y|^2 \, dy < 20\,\delta x^2 \log^2 x.$$

Now we employ an explicit version of a result by Goldston (1983):

Theorem (EHKMMT 2022)

For any $z > x > 10^{18}$ we have

$$\psi(x) = x - \sum_{|\Im\rho|<z} \frac{x^\rho}{\rho} + O^*(5\log x \log\log x),$$

where the sum is over the nontrivial zeros of the Riemann zeta function (counted with multiplicity).

Then it remains to show that

$$\int_x^{2x} \left| \sum_{|\Im\rho|<3x} y^\rho C(\rho) \right|^2 dy < 9.942\delta x^2 \log^2 x,$$

where

$$C(\rho) := \frac{1-(1+\delta)^\rho}{\rho}.$$

Here the calculation becomes technical. In big steps:

$$\begin{aligned} \text{LHS} &< \int_1^2 \int_{xv/2}^{2xv} \left| \sum_{|\Im\rho|<3x} y^\rho C(\rho) \right|^2 dy\, dv \\ &< x^2 \sum_{\rho,\rho'} |C(\rho)|^2 \left| \frac{2^2+2^{-2}}{2+\rho-\rho'} \right| \left| \frac{2^3+1}{3+\rho-\rho'} \right| \\ &< 15.616 x^2 \sum_{\Im\rho>0} \min\left(\delta^2, \frac{4}{(\Im\rho)^2}\right) \left(\frac{1}{2} + \log\frac{\Im\rho}{2\pi}\right). \end{aligned}$$

Without the Riemann hypothesis, we are unable to prove Zoli's conjectures. However, we can verify them for sufficiently large n, proving that the DPG-process creates an infinite sequence of prime gap graphs.

Regarding the first conjecture, we recall

Theorem (Erdős–Gallai 1960)

Let $d_1 \geqslant \cdots \geqslant d_n \geqslant 0$ be integers. Then the sequence $(d_1, \ldots, d_n)$ is graphic if and only if $d_1 + \cdots + d_n$ is even and for every $k \in \{1, \ldots, n\}$ we have

$$\sum_{\ell=1}^{k} d_\ell \leqslant k(k-1) + \sum_{\ell=k+1}^{n} \min(k, d_\ell) \, .$$

Interestingly, we can apply this result to a long initial segment of the prime gap sequence even though this sequence is not ordered.

Graphicality of the prime gap sequence without RH (2 of 5)

Theorem (EHKMMT 2022)

Let $\mathbf{D} = (d_1, \ldots, d_n)$ be a sequence of positive integers such that $\|\mathbf{D}\|_1 = \sum_{\ell=1}^{n} d_\ell$ is even. Let $1 < p \leqslant \infty$ be a parameter, and assume that the following L^p-norm bound holds:

$$\|2 + \mathbf{D}\|_p \leqslant n^{\frac{1}{2} + \frac{1}{2p}}.$$

Then there is a simple graph G with degree sequence $\mathbf{D}$.

Proof (sketch).

By symmetry, we can assume that $d_1 \geqslant \cdots \geqslant d_n$. Denoting $\mathbf{D}^k := (d_1, \ldots, d_k)$, we strengthen the Erdős–Gallai condition to $\|2 + \mathbf{D}^k\|_1 \leqslant k^2 + n$. This stronger condition follows from the initial assumption and Hölder's inequality, hence we are done:

$$\|2 + \mathbf{D}^k\|_1 \leqslant k^{1-\frac{1}{p}} \|2 + \mathbf{D}^k\|_p \leqslant k^{1-\frac{1}{p}} n^{\frac{1}{2} + \frac{1}{2p}} < k^2 + n. \qquad \square$$

The previous theorem reduces Zoli's first conjecture to

$$\sum_{\ell=1}^{n}(2+p_\ell-p_{\ell-1})^2 \leqslant n^{3/2}.$$

As Heath-Brown (1978) proved that the left-hand side is at most $n^{4/3+o(1)}$, the conjecture indeed holds for every sufficiently large n.

For Zoli's second conjecture, we developed

Theorem (EHKMMT 2022)

Let $\mathbf{D}=(d_1,\ldots,d_n)$ be a sequence of positive integers such that $\|\mathbf{D}\|_1=\sum_{\ell=1}^{n} d_\ell$ is even. Let $1<p\leqslant\infty$ be a parameter, and let G be any simple graph with degree sequence $\mathbf{D}$. Assume that $d\geqslant 2$ is an even integer satisfying

$$4d^{1-\frac{1}{p}}\|\mathbf{D}\|_p \leqslant \|\mathbf{D}\|_1.$$

Then G contains $d/2$ independent edges.

Proof (sketch).

By Vizing's theorem, it suffices to verify that the following condition holds for some integer $\delta \geqslant 1$:

$$\frac{1}{\delta}\left(\frac{1}{2}\sum_{\ell=1}^{n} d_\ell - \sum_{d_\ell \geqslant \delta} d_\ell\right) \geqslant \frac{d}{2}.$$

If $p = \infty$, then we can choose $\delta := 1 + \|\mathbf{D}\|_\infty$. So let us focus on the case $1 < p < \infty$. For any integer $\delta \geqslant 1$, we have

$$\sum_{\ell=1}^{n} d_\ell - 2\sum_{d_\ell \geqslant \delta} d_\ell \geqslant \|\mathbf{D}\|_1 - 2\delta^{1-p}\|\mathbf{D}\|_p^p,$$

hence it suffices that

$$\delta^{1-p}\|\mathbf{D}\|_p^p \leqslant \frac{1}{4}\|\mathbf{D}\|_1 \qquad \text{and} \qquad \delta d \leqslant \frac{1}{2}\|\mathbf{D}\|_1.$$

Graphicality of the prime gap sequence without RH (5 of 5)

Proof (sketch, continued).

In other words, it suffices to find an integer δ satisfying

$$\left(\frac{4\|\mathbf{D}\|_p^p}{\|\mathbf{D}\|_1}\right)^{\frac{1}{p-1}} \leqslant \delta \leqslant \frac{1}{2d}\|\mathbf{D}\|_1.$$

The left-hand side exceeds 1, hence δ exists as long as

$$2\left(\frac{4\|\mathbf{D}\|_p^p}{\|\mathbf{D}\|_1}\right)^{\frac{1}{p-1}} \leqslant \frac{1}{2d}\|\mathbf{D}\|_1. \qquad \square$$

The previous theorem reduces Zoli's second conjecture to

$$16(p_{n+1}-p_n)\sum_{\ell=1}^{n}(p_\ell - p_{\ell-1})^2 \leqslant (p_n - 1)^2.$$

By Ingham (1937) and Heath-Brown (1978), the left-hand side is at most $n^{5/8+4/3+o(1)} = n^{47/24+o(1)}$, so we are done for large n.