The sequence of prime gaps is graphic

(joint with P. L. Erdős, S. R. Kharel, P. Maga, T. R. Mezei, Z. Toroczkai)

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20 April 2022
Shandong University
Number Theory Seminar

Introducing prime gap graphs

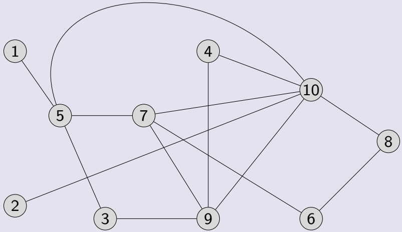
Definition

Let p_n denote the n-th prime number, and let $p_0 = 1$.

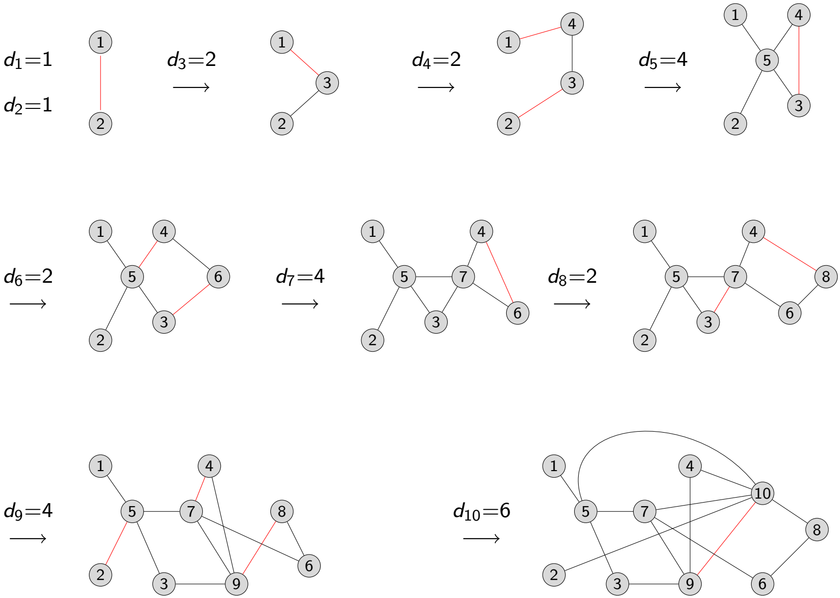
We call a simple graph on $n \geqslant 2$ vertices a *prime gap graph* if its vertex degrees are $p_1 - p_0, \ldots, p_n - p_{n-1}$.

Example ($n = 10$)

1	2	3	5	7	11	13	17	19	23	29
	1	1	2	2	4	2	4	2	4	6



Prime gap graphs generated by a DPG-process



Imagine that we made to 30 vertices...

- A gap of 14 occurs between $p_{30} = 113$ and $p_{31} = 127$. The earlier gaps are smaller, in fact they do not exceed 8.
- Imagine that we made to 30 vertices with the DPG-process. Then our prime gap graph has $(p_{30} - 1)/2 = 56$ edges.
- To continue, we want to remove $14/2 = 7$ independent edges, and connect their 14 ends to a new vertex, creating a prime gap graph with 31 vertices and $(p_{31} - 1)/2 = 63$ edges.
- How can we guarantee $14/2 = 7$ independent edges without actually looking at the graph?

Theorem (Vizing 1964)

The edges of a simple graph with maximal vertex degree Δ can be colored with $\Delta + 1$ colors.

Corollary

The 56 edges of a prime gap graph on 30 vertices can be colored with 9 colors. The largest color class has at least 7 members, because $9 \cdot 6 < 56$, and it consists of independent edges.

Main conjectures and main theorem

Conjecture (Toroczkai 2016)

For every $n \geqslant 2$, there exists a prime gap graph on n vertices.

Conjecture (Toroczkai 2016)

In every prime gap graph on n vertices, there exist $(p_{n+1} - p_n)/2$ independent edges.

Remark

The second conjecture says that, starting with the prime gap graph on 2 vertices, the DPG-process runs indefinitely. Hence it implies the first conjecture.

Theorem

The above conjectures are true for every sufficiently large n. Assuming the Riemann hypothesis, they are true for every $n \geqslant 2$.

$(p_{n+1} - p_n)/2$ independent edges under RH (1 of 2)

We can assume $n \geqslant 5$. By Vizing's theorem, it suffices that

$$\left\lceil \frac{p_n - 1}{2N} \right\rceil \geqslant \frac{p_{n+1} - p_n}{2},$$

where

$$N := \max_{1 \leqslant m \leqslant n} (1 + p_m - p_{m-1}).$$

For $5 \leqslant n \leqslant 44$ we checked this by a simple computer program. For $n \geqslant 45$, it would suffice to prove the following

Conjecture (cf. Oppermann 1877 & Legendre 1797)

For any $x \geqslant 117$, there is a prime number in $[x, x + \sqrt{x}]$.

Indeed, let $k \geqslant 15$ be the integer satisfying $(k-1)^2 < p_n < k^2$. The conjecture implies that $p_{n+1} - p_n \leqslant k - 1$ and $N \leqslant k$, hence

$$\left\lceil \frac{p_n - 1}{2N} \right\rceil \geqslant \frac{k-1}{2} \geqslant \frac{p_{n+1} - p_n}{2}.$$

$(p_{n+1} - p_n)/2$ independent edges under RH (2 of 2)

The previous conjecture holds for $x \in [117, 10^{18}]$ by known prime gap records, hence we can assume that $p_n > 10^{18}$. We shall use:

Theorem (Carneiro–Milinovich–Soundararajan 2019)

Assume the Riemann hypothesis. Then, for any $x \geqslant 4$, there is a prime number in $[x, x + \frac{22}{25}\sqrt{x}\log x]$.

Theorem (EHKMMT 2022)

Assume the Riemann hypothesis. Let $p_n > 10^{18}$, and let G be a prime gap graph on n vertices. For at least one-third of the edges of G, the endpoints have degrees less than

$$N := \left\lceil \frac{\sqrt{p_n}}{3\log p_n} \right\rceil .$$

We succeed by applying Vizing's theorem to a subgraph of G:

$$\left\lceil \frac{p_n - 1}{6N} \right\rceil > 0.499\sqrt{p_n}\log p_n > \frac{p_{n+1} - p_n}{2}.$$

Main analytic input under RH (1 of 3)

The last theorem follows from an explicit version of a result by Selberg (1943):

Theorem (EHKMMT 2022)

Assume the Riemann hypothesis. Then, for any $x \geqslant 2$ and $N > 0$, we have

$$\sum_{\substack{x \leqslant p_m \leqslant 2x \\ p_{m+1} - p_m \geqslant N}} (p_{m+1} - p_m) < \frac{163 x \log^2 x}{N}.$$

Indeed, for

$$p_n > 10^{18} \qquad \text{and} \qquad N := \left\lceil \frac{\sqrt{p_n}}{3 \log p_n} \right\rceil$$

we obtain

$$\sum_{\substack{m \leqslant n \\ p_m - p_{m-1} \geqslant N}} (p_m - p_{m-1}) < 489 \sqrt{p_n} \log^3 p_n < \frac{p_n - 1}{3}.$$

Main analytic input under RH (2 of 3)

The proof relies on ideas of Heath-Brown (1978) and Saffari–Vaughan (1977). First, one can restrict to $x > 10^{18}$ and

$$81 \log^2 x < N < \frac{4}{3}\sqrt{x} \log x.$$

Then, writing $N = 4\delta x$, the statement can be reduced to

$$\int_x^{2x} |\psi(y + \delta y) - \psi(y) - y\delta|^2 \, dy < 20 \, \delta x^2 \log^2 x.$$

Now we employ an explicit version of a result by Goldston (1983):

Theorem (EHKMMT 2022)

For any $z > x > 10^{18}$ we have

$$\psi(x) = x - \sum_{|\Im\rho|<z} \frac{x^\rho}{\rho} + O^*(5 \log x \log\log x),$$

where the sum is over the nontrivial zeros of the Riemann zeta function (counted with multiplicity).

Main analytic input under RH (3 of 3)

Then it remains to show that

$$\int_x^{2x} \left| \sum_{|\Im\rho|<3x} y^\rho C(\rho) \right|^2 \, dy < 9.942\delta x^2 \log^2 x,$$

where

$$C(\rho) := \frac{1-(1+\delta)^\rho}{\rho}.$$

Here the calculation becomes technical. In big steps:

$$\begin{aligned}
\text{LHS} &< \int_1^2 \int_{xv/2}^{2xv} \left| \sum_{|\Im\rho|<3x} y^\rho C(\rho) \right|^2 \, dy \, dv \\
&< x^2 \sum_{\rho,\rho'} |C(\rho)|^2 \left| \frac{2^2+2^{-2}}{2+\rho-\rho'} \right| \left| \frac{2^3+1}{3+\rho-\rho'} \right| \\
&< 15.616 x^2 \sum_{\Im\rho>0} \min\left(\delta^2, \frac{4}{(\Im\rho)^2}\right) \left(\frac{1}{2} + \log \frac{\Im\rho}{2\pi}\right).
\end{aligned}$$

Graphicality of the prime gap sequence without RH (1 of 5)

Without the Riemann hypothesis, we are unable to prove Zoli's conjectures. However, we can verify them for sufficiently large n, proving that the DPG-process creates an infinite sequence of prime gap graphs.

Regarding the first conjecture, we recall

Theorem (Erdős–Gallai 1960)

Let $d_1 \geqslant \cdots \geqslant d_n \geqslant 0$ be integers. Then the sequence $(d_1, \ldots, d_n)$ is graphic if and only if $d_1 + \cdots + d_n$ is even and for every $k \in \{1, \ldots, n\}$ we have

$$\sum_{i=1}^{k} d_i \leqslant k(k-1) + \sum_{i=k+1}^{n} \min(k, d_i) .$$

Interestingly, we can apply this result to a long initial segment of the prime gap sequence even though this sequence is not ordered.

Graphicality of the prime gap sequence without RH (2 of 5)

Theorem (EHKMMT 2022)

Let $\mathbf{D} = (d_1, \ldots, d_n)$ *be a sequence of positive integers such that* $\|\mathbf{D}\|_1 = \sum_{i=1}^n d_i$ *is even. Let* $1 < p \leqslant \infty$ *be a parameter, and assume that the following* L^p*-norm bound holds:*

$$\|2 + \mathbf{D}\|_p \leqslant n^{\frac{1}{2} + \frac{1}{2p}}.$$

Then there is a simple graph G *with degree sequence* $\mathbf{D}$*.*

Proof (sketch).

By symmetry, we can assume that $d_1 \geqslant \cdots \geqslant d_n$. Denoting $\mathbf{D}^k := (d_1, \ldots, d_k)$, we strengthen the Erdős–Gallai condition to $\|2 + \mathbf{D}^k\|_1 \leqslant k^2 + n$. This stronger condition follows from the initial assumption and Hölder's inequality, hence we are done:

$$\|2 + \mathbf{D}^k\|_1 \leqslant k^{1-\frac{1}{p}} \|2 + \mathbf{D}^k\|_p \leqslant k^{1-\frac{1}{p}} n^{\frac{1}{2} + \frac{1}{2p}} < k^2 + n. \qquad \square$$

Graphicality of the prime gap sequence without RH (3 of 5)

The previous theorem reduces Zoli's first conjecture to

$$\sum_{i=1}^{n}(2 + p_i - p_{i-1})^2 \leqslant n^{3/2}.$$

As Heath-Brown (1978) proved that the left-hand side is at most $n^{4/3+o(1)}$, the conjecture indeed holds for every sufficiently large n.

For Zoli's second conjecture, we developed

Theorem (EHKMMT 2022)

Let $\mathbf{D} = (d_1, \ldots, d_n)$ be a sequence of positive integers such that $\|\mathbf{D}\|_1 = \sum_{i=1}^{n} d_i$ is even. Let $1 < p \leqslant \infty$ be a parameter, and let G be any simple graph with degree sequence $\mathbf{D}$. Assume that $d \geqslant 2$ is an even integer satisfying

$$4d^{1-\frac{1}{p}}\|\mathbf{D}\|_p \leqslant \|\mathbf{D}\|_1.$$

Then G contains $d/2$ independent edges.

Graphicality of the prime gap sequence without RH (4 of 5)

Proof (sketch).

By Vizing's theorem, it suffices to verify that the following condition holds for some integer $\delta \geqslant 1$:

$$\frac{1}{\delta}\left(\frac{1}{2}\sum_{i=1}^{n} d_i - \sum_{d_i \geqslant \delta} d_i\right) \geqslant \frac{d}{2}.$$

If $p = \infty$ we can choose $\delta := 1 + \|\mathbf{D}\|_\infty$. So let us focus on the case $1 < p < \infty$. For any integer $\delta \geqslant 1$, we have

$$\sum_{i=1}^{n} d_i - 2\sum_{d_i \geqslant \delta} d_i \geqslant \|\mathbf{D}\|_1 - 2\delta^{1-p}\|\mathbf{D}\|_p^p,$$

hence it suffices that

$$\delta^{1-p}\|\mathbf{D}\|_p^p \leqslant \frac{1}{4}\|\mathbf{D}\|_1 \qquad \text{and} \qquad \delta d \leqslant \frac{1}{2}\|\mathbf{D}\|_1.$$

Proof (sketch, continued).

In other words, it suffices to find an integer δ satisfying

$$\left(\frac{4\|\mathbf{D}\|_p^p}{\|\mathbf{D}\|_1}\right)^{\frac{1}{p-1}} \leqslant \delta \leqslant \frac{1}{2d}\|\mathbf{D}\|_1.$$

The left-hand side exceeds 1, hence δ exists as long as

$$2\left(\frac{4\|\mathbf{D}\|_p^p}{\|\mathbf{D}\|_1}\right)^{\frac{1}{p-1}} \leqslant \frac{1}{2d}\|\mathbf{D}\|_1.$$

□

The previous theorem reduces Zoli's second conjecture to

$$16(p_{n+1}-p_n)\sum_{i=1}^{n}(p_i-p_{i-1})^2 \leqslant (p_n-1)^2.$$

By Ingham (1937) and Heath-Brown (1978), the left-hand side is at most $n^{5/8+4/3+o(1)} = n^{47/24+o(1)}$, so we are done for large n.