The subconvexity problem for Rankin-Selberg L-functions and equidistribution of Heegner points

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Established cases of subconvexity over $\mathbb{Q}$

s: point on critical line ($\Re s = \frac{1}{2}$)
χ: primitive Dirichlet character modulo q
f: holomorphic cusp form of full level
g: Maass cusp form of full level

$$\zeta(s) \ll_\epsilon |s|^{\frac{1}{4}-\frac{1}{12}+\epsilon}$$ **Weyl 1921**

$$L(s,\chi) \ll_{\epsilon,s} q^{\frac{1}{4}-\frac{1}{16}+\epsilon}$$ **Burgess 1963**

$$L(s,\chi) \ll_\epsilon (|s|q)^{\frac{1}{4}-\frac{1}{16}+\epsilon}$$ **HB 1980**

$$L(s,f) \ll_{\epsilon,f} |s|^{\frac{1}{2}-\frac{1}{6}+\epsilon}$$ **Good 1982**

$$L(s,g) \ll_{\epsilon,g} |s|^{\frac{1}{2}-\frac{1}{6}+\epsilon}$$ **Me 1990**

χ: primitive Dirichlet character of conductor q
g: primitive holomorphic or Maass cusp form
s: point on critical line ($\Re s = \frac{1}{2}$)

$$L(s, \chi \otimes g) \ll_{s,g} q^{\frac{1}{2}-\delta}$$

- $\delta < \frac{1}{22}$ for g holomorphic of full level (Duke–Friedlander–Iwaniec, 1993)

- $\delta < \frac{1}{8}$ for g holomorphic (Bykovskiĭ, 1996)

- $\delta < \frac{1}{54}$ (Harcos, 2001)

- $\delta < \frac{1}{22}$ (Michel, 2002)

- $\delta < \frac{1-2\theta}{10+4\theta}$ under Hypothesis H_θ (Blomer, 2004)

- $\delta < \frac{1-2\theta}{8}$ under Hypothesis H_θ (Blomer–Harcos–Michel, 2004)

Hypothesis H_θ**.** *For any cuspidal automorphic form* π *on* GL_2 *over* $\mathbb{Q}$ *with local Hecke parameters* $\alpha_\pi^{(1)}(p),\ \alpha_\pi^{(2)}(p)$ *for* $p < \infty$ *and* $\mu_\pi^{(1)}(\infty),\ \mu_\pi^{(2)}(\infty)$*, one has the bounds*

$$|\alpha_\pi^{(j)}(p)| \leqslant p^\theta, \text{ if } \pi_p \text{ is unramified;}$$
$$|\Re\mu_\pi^{(j)}(\infty)| \leqslant \theta, \quad \text{if } \pi_\infty \text{ is unramified.}$$

- H_0 is the classical Ramanujan–Selberg conjecture
- $H_{\frac{7}{64}}$ was proved by Kim–Sarnak–Shahidi (2003)

f: holomorphic cusp form of weight k and full level

$$L(s,f) \ll_\epsilon (|s|+k)^{\frac{1}{2}-\frac{1}{6}+\epsilon} \quad \textbf{Pe–Ju–Mo 2001–2004}$$

f: Maass cusp form of eigenvalue $\frac{1}{4}+t^2$ and full level

$$L(s,f) \ll_\epsilon (|s|+|t|)^{\frac{1}{2}-\frac{1}{6}+\epsilon} \quad \textbf{Iv–Ju–Mo 2001–2004}$$

f: holomorphic or Maass cusp form of level q, parameter t, and trivial or primitive nebentypus

$$L(s,f) \ll_{\epsilon,s,t} q^{\frac{1}{4}-\frac{1}{23040}+\epsilon} \quad \textbf{Du–Fr–Iw 1993–2002}$$

f: holomorphic cusp form of weight k and level q
g: holomorphic or Maass cusp form

$$L(s, f \otimes g) \ll_{\epsilon,s,g,q} k^{1-\frac{25}{601}+\epsilon} \quad \textbf{Sarnak 2001}$$

f: Maass cusp form of eigenvalue $\frac{1}{4} + t^2$ and level q
g: holomorphic or Maass cusp form

$$L(s, f \otimes g) \ll_{\epsilon,s,g,q} |t|^{1-\frac{25}{512}+\epsilon} \quad \textbf{Liu–Ye 2002–2004}$$

f: holomorphic or Maass cusp form of level q and parameter t
g: holomorphic cusp form

$$L(s, f \otimes g) \ll_{\epsilon,s,g,t} q^{\frac{1}{2}-\frac{1}{1057}+\epsilon} \quad \textbf{Ko–Mi–Va 2001–2002}$$

f: holomorphic or Maass cusp form of level q and parameter t
g: Maass cusp form

$$L(s, f \otimes g) \ll_{\epsilon,s,g,t} q^{\frac{1}{2}-\frac{1}{2648}+\epsilon} \quad \textbf{Ha–Mi 2004}$$

The problem

f: primitive cusp form of level q, nebentypus χ_f, eigenvalue $\frac{1}{4}+t_f^2$
g: primitive cusp form of level D, nebentypus χ_g, eigenvalue $\frac{1}{4}+t_g^2$
s: point on critical line ($\Re s = \frac{1}{2}$)

$$L(s, f \otimes g) \overset{?}{\ll} q^{\frac{1}{2}-\delta}$$

Why do we care?

K: imaginary quadratic number field of discriminant $-q$
$\mathcal{O}_K$: ring of integers of K
$\mathrm{Ell}(\mathcal{O}_K)$: set of elliptic curves with complex multiplication by $\mathcal{O}_K$

- defined over H_K, the Hilbert class field of K
- corresponds to Heegner points on $\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H}$
- actions by $\mathrm{Gal}(H_K/K)$ and $\mathrm{Pic}(\mathcal{O}_K)$ agree via Artin map
- $|\,\mathrm{Pic}(\mathcal{O}_K)| \gg_\varepsilon q^{\frac{1}{2}-\varepsilon}$ by Siegel's theorem

Equidistribution

z: Heegner point for K of discriminant $-q$
G: subgroup of $G_K := \mathrm{Gal}(H_K/K)$
Gz: short or orbit of Heegner points for K of discriminant $-q$
g: primitive Maass cusp form or $E\left(., \frac{1}{2} + it\right)$ on $\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H}$

$$\frac{1}{|G|} \sum_{\sigma \in G} g(z^\sigma) = \frac{1}{|G_K|} \sum_{\substack{\psi \in \widehat{G}_K \\ \psi_{|G} \equiv 1}} \sum_{\sigma \in G_K} \psi(\sigma) g(z^\sigma)$$

Zhang (2001):
$$\left| \sum_{\sigma \in G_K} \psi(\sigma) g(z^\sigma) \right|^2 = \frac{|\mathcal{O}_K^\times| \sqrt{q}}{4} L\left(\tfrac{1}{2}, f_\psi \otimes g\right)$$

f_ψ: holomorphic theta series of weight 1 and level q associated to the character ψ

- If ψ factors through the norm $N_{K/\mathbf{Q}}$ then f_ψ is an Eisenstein series:

  $$L\Big(\tfrac{1}{2}, f_\psi \otimes g\Big) = L\Big(\tfrac{1}{2}, \chi_1 \otimes g\Big) L\Big(\tfrac{1}{2}, \chi_2 \otimes g\Big),$$

  where χ_1, χ_2 are Dirichlet characters such that $\chi_1 \chi_2 = \left(\frac{-q}{\cdot}\right)$.

- Otherwise, f_ψ is a cusp form.

Previous versions

$$L(s, f \otimes g) \ll q^{\frac{1}{2}-\delta}$$

- Kowalski, Michel, VanderKam (2001): f holomorphic, conductor of $\chi_f \chi_g$ is at most $q^{\frac{1}{2}-\eta}$, g holomorphic or non-exceptional Maass, D square-free

- Michel (2002): g holomorphic, $\delta = \frac{1}{1057}$

New version (joint work with Michel)

$$\mathrm{cond}(\chi_f \chi_g) > 1$$

$$\theta := \frac{7}{64}, \qquad \delta_{\mathrm{tw}} := \frac{1-2\theta}{8}$$

$$L(s, f \otimes g) \ll_{\varepsilon,s,g} q^{\varepsilon + \frac{1}{2} - \frac{(1-2\theta)\delta_{\mathrm{tw}}}{202}}$$

Theorem. *For any bounded and uniformly continuous function* $g : \mathsf{SL}_2(\mathbf{Z})\backslash\mathbf{H} \to \mathbf{C}$*, there exists a bounded function* $\varepsilon_g : \mathbf{R}^+ \to \mathbf{R}^+$*, depending only on* g*, which satisfies*

$$\lim_{x\to 0} \varepsilon_g(x) = 0$$

such that: for any imaginary quadratic field K *with discriminant* $-q$*, any subgroup* $G \subset G_K$*, and any* $E \in \mathsf{Ell}(\mathcal{O}_K)$*, one has*

$$\left|\frac{1}{|G|}\sum_{\sigma\in G} g(\varphi(E^\sigma)) - \int_{\mathsf{SL}_2(\mathbf{Z})\backslash\mathbf{H}} g(z)\frac{3}{\pi}\frac{dxdy}{y^2}\right| \leqslant \varepsilon_g\Big([G_K : G]q^{-\frac{1}{23042}}\Big).$$

Main ingredients

- approximate functional equation
- amplification
- Kuznetsov formula (forwards and backwards)
- Voronoi summation formula (twice)
- circle method (Jutila's variant)
- subconvexity for $L(s,\chi)$, $L(s,\chi\otimes g)$
- uniform bounds for exponential sums associated with cusp forms
- uniform bounds for Bessel functions

Rankin-Selberg L-functions

$$L(s,f)=\sum_{n\geqslant 1}\frac{\lambda_f(n)}{n^s},\quad L(s,g)=\sum_{n\geqslant 1}\frac{\lambda_g(n)}{n^s}$$

$$L(s,f\otimes g)\approx\sum_{n\geqslant 1}\frac{\lambda_f(n)\lambda_g(n)}{n^s}$$

$$\Lambda(s,f\otimes g)=Q(f\otimes g)^{s/2}L_\infty(s,f\otimes g)L(s,f\otimes g)$$

Conductor of $f\otimes g$:

$$\frac{(qD)^2}{(q,D)^4}\leqslant Q(f\otimes g)\leqslant\frac{(qD)^2}{(q,D)}$$

Approximate functional equation

$$\mu_f := 1 + |t_f|, \quad \mu_g := 1 + |t_g|, \quad P := (|s| + \mu_f + \mu_g)^2$$

$$L(s, f \otimes g) \ll_A \log^2(qDP + 1) \sum_{N=2^\nu} \frac{\left|L_{f\otimes g}(N)\right|}{\sqrt{N}} \left(1 + \frac{N}{qDP}\right)^{-A}$$

$$L_{f\otimes g}(N) = \sum_n \lambda_f(n)\lambda_g(n)W(n)$$

$$\operatorname{supp} W \subset [N/2, 2N], \quad x^j W^{(j)}(x) \ll_{j,A} P^j$$

Amplification

L: small power of q

$\vec{x}$: complex sequence (x_ℓ) supported on $\{\ell \leqslant L : (\ell, qD) = 1\}$

$$L_{f\otimes g}(\vec{x}, N) := \rho_f(1)\left(\sum_{\ell \leqslant L} x_\ell \lambda_f(\ell)\right) L_{f\otimes g}(N)$$

$$= \sum_{\ell} x_\ell \sum_{de=\ell} \chi_f(d) \sum_{ab=d} \mu(a)\chi_g(a)\lambda_g(b) \sum_{n} W(adn)\lambda_g(n)\sqrt{aen}\rho_f(aen).$$

“Spectrally complete” quadratic form:

$$Q(\vec{x},N) := \sum_j \mathcal{H}(t_j)\Big|L_{u_j\otimes g}(\vec{x},N)\Big|^2 + \sum_{\mathfrak{a}} \frac{1}{4\pi}\int_{\mathbf{R}} \mathcal{H}(t)\Big|L_{\mathfrak{a},t,g}(\vec{x},N)\Big|^2 dt,$$

where for $u \in \mathcal{L}_k([q,D],\chi_f)$:

$$L_{u\otimes g}(\vec{x},N) := \sum_\ell x_\ell \sum_{de=\ell} \chi_f(d) \sum_{ab=d} \mu(a)\chi_g(a)\lambda_g(b) \sum_n W(adn)\lambda_g(n)\sqrt{aen}\rho_u(aen),$$

and $\mathcal{H}(t) : \mathbf{R}\cup i\mathbf{R} \to (0,\infty)$ is a weight function enabling a pleasant variant of Kuznetsov’s formula.

Spectral decomposition of $\mathcal{L}_k([q,D],\chi_f)$:

$$u_j(z) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \rho_j(n) W_{\frac{n}{|n|}\frac{k}{2},it}(4\pi|n|y)e(nx)$$

$$E_{\mathfrak{a}}\Big(z,\tfrac{1}{2}+it\Big) = \delta_{\mathfrak{a}\infty} y^{\frac{1}{2}+it} + \phi_{\mathfrak{a}}\Big(\tfrac{1}{2}+it\Big) y^{\frac{1}{2}-it} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \rho_{\mathfrak{a}}(n,t) W_{\frac{n}{|n|}\frac{k}{2},it}(4\pi|n|y)e(nx)$$

$W_{\alpha,\beta}(y)$: Whittaker function

$\phi_{\mathfrak{a}}\Big(\frac{1}{2}+it\Big)$: entry $(\infty,\mathfrak{a})$ of the scattering matrix.

Proposition. *For any integer $k \geqslant 0$ and any $A > 0$, there exist functions $\mathcal{H}(t) : \mathbf{R} \cup i\mathbf{R} \to (0, \infty)$ and $\mathcal{I}(x) : (0, \infty) \to \mathbf{R} \cup i\mathbf{R}$ depending only on k and A such that*

$$\mathcal{H}(t) \gg_A (1 + |t|)^{k-16} e^{-\pi|t|};$$

for any integer $j \geqslant 0$,

$$x^j \mathcal{I}^{(j)}(x) \ll_{A,j} \left(\frac{x}{1+x}\right)^{A+1} (1+x)^{1+j};$$

and for any positive integers m, n,

$$\sqrt{mn} \sum_{j \geqslant 1} \mathcal{H}(t_j) \overline{\rho}_j(m) \rho_j(n) + \sqrt{mn} \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{\mathbf{R}} \mathcal{H}(t) \overline{\rho}_{\mathfrak{a}}(m, t) \rho_{\mathfrak{a}}(n, t) dt$$

$$= c_A \delta_{m,n} + \sum_{c \equiv 0\,([q,D])} \frac{S_{\chi_f}(m, n; c)}{c} \mathcal{I}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

Standard amplifier $\vec{x} = (x_1, \ldots, x_L)$:

$$x_\ell := \begin{cases} \lambda_f(p)\overline{\chi}_f(p) & \text{if} \quad \ell = p,\ (p, qD) = 1,\ \sqrt{L}/2 < p \leqslant \sqrt{L}; \\ -\overline{\chi}_f(p) & \text{if} \quad \ell = p^2,\ (p, qD) = 1,\ \sqrt{L}/2 < p \leqslant \sqrt{L}; \\ 0 & \text{else.} \end{cases}$$

$$\forall p : \lambda_f(p)^2 - \lambda_f(p^2) = \chi_f(p) \quad \Rightarrow \quad \left| \sum_{\ell \leqslant L} x_\ell \lambda_f(\ell) \right| \gg_\varepsilon q^{-\varepsilon}\sqrt{L}$$

$$\left| \sum_{\ell \leqslant L} x_\ell \lambda_f(\ell) \right|^2 \left| L_{f \otimes g}(N) \right|^2 \ll_\varepsilon q^{1+\varepsilon} Q(\vec{x}, N)$$

$$\|\vec{x}\|_1 + \|\vec{x}\|_2^2 \ll_\varepsilon q^\varepsilon \sqrt{L}$$

$$N \leqslant (qDP)^{1+\varepsilon}, \qquad \|\vec{x}\|_1 := \sum_{\ell \leqslant L} |x_\ell|, \qquad \|\vec{x}\|_2^2 := \sum_{\ell \leqslant L} |x_\ell|^2$$

$$Q(\vec{x}, N) \ll_\varepsilon q^\varepsilon N \left\{ \|\vec{x}\|_2^2 + \|\vec{x}\|_1^2 \left(L^{\delta_L} q^{-\delta_q} + L^{\delta_{3L}} q^{-\delta_{3q}} + L^{\delta_{4L}} q^{-\delta_{4q}} \right) \right\}$$

$$\delta_L := \frac{46 - 9\theta - 22\theta^2}{9} \qquad \delta_q := \frac{1 - 2\theta}{9}\, \delta_{\mathrm{tw}}$$

$$\delta_{3L} := 9 + 4\theta \qquad \delta_{3q} := \frac{1}{2} - \theta$$

$$\delta_{4L} := \frac{7 + 10\theta + 4\theta^2}{2(1+\theta)} \qquad \delta_{4q} := \frac{1 + 4\theta}{4(1+\theta)}$$

Analysis of the quadratic form

- $Q(\vec{x}, N)$ decomposes into diagonal and non-diagonal parts

- diagonal part consists of sums of the form

$$\sum_{a_1 e_1 m = a_2 e_2 n} \overline{\lambda_g}(m) \lambda_g(n) \overline{W}(a_1 d_1 m) W(a_2 d_2 n)$$

- non-diagonal part consists of sums of the form

$$\sum_{m,n} \overline{\lambda_g}(m) \lambda_g(n) S_{\chi_f}(a_1 e_1 m, a_2 e_2 n; c) \times \mathcal{I}\left(\frac{4\pi\sqrt{a_1 a_2 e_1 e_2 m n}}{c}\right) \overline{W}(a_1 d_1 m) W(a_2 d_2 n)$$

Voronoi summation formula

Proposition. *Let* $c \equiv 0\,(D)$ *and* a *be an integer coprime to* c. *If* $F \in C^\infty(\mathbf{R}^{\times,+})$ *is a Schwartz class function vanishing in a neighborhood of zero, then*

$$\sum_{n \geqslant 1} \sqrt{n}\rho_g(n) e\Big(n\frac{a}{c}\Big) F(n) = \frac{\chi(\overline{a})}{c} \sum_{\pm} \sum_{n \geqslant 1} \sqrt{n}\rho_g^{\pm}(n) e\Big(\mp n\frac{\overline{a}}{c}\Big) \mathcal{F}^{\pm}\Big(\frac{n}{c^2}\Big).$$

In this formula,

$$\rho_g^{+}(n) := \rho_g(n), \quad \rho_g^{-}(n) := \rho_{Qg}(n) = \frac{\Gamma\Big(\frac{1}{2} + it - \frac{k}{2}\Big)}{\Gamma\Big(\frac{1}{2} + it + \frac{k}{2}\Big)} \rho_g(-n),$$

and

$$\mathcal{F}^{\pm}(y) := \int_0^\infty F(x) J_g^{\pm}\Big(4\pi\sqrt{xy}\Big)\, dx,$$

where

- $$J_g^+(x) := 2\pi i^l J_{l-1}(x), \;\; J_g^-(x) := 0,$$
  if g is induced from a holomorphic form of weight l;

- $$J_g^+(x) := \frac{-\pi}{\mathrm{ch}(\pi t)}\Big\{Y_{2it}(x)+Y_{-2it}(x)\Big\}, \;\; J_g^-(x) := 4\,\mathrm{ch}(\pi t)K_{2it}(x),$$
  if k is even, and g is not induced from a holomorphic form;

- $$J_g^+(x) := \frac{\pi}{\mathrm{sh}(\pi t)}\Big\{Y_{2it}(x)-Y_{-2it}(x)\Big\}, \;\; J_g^-(x) := -4i\,\mathrm{sh}(\pi t)K_{2it}(x),$$
  if k is odd, and g is not induced from a holomorphic form.

A shifted convolution problem

g: primitive cusp form of level D and nebentypus χ_g
χ: primitive character of modulus $q > 1$
$G_\chi(h; \mathbf{c})$: Gauss sum of the (induced) character χ (mod $\mathbf{c}$)
ℓ_1, ℓ_2: two positive integers
$F(x,y)$: a smooth function supported on $[X/4, 4X] \times [Y/4, 4Y]$

$$x^i y^j \frac{\partial^i}{\partial x^i} \frac{\partial^j}{\partial y^j} F(x,y) \ll Z^{i+j}$$

$$\Sigma_\chi^{\pm}(\ell_1, \ell_2; \mathbf{c}) := \sum_{h \neq 0} G_\chi(h; \mathbf{c}) S_h^{\pm}(\ell_1, \ell_2)$$

$$S_h^{\pm}(\ell_1, \ell_2) := \sum_{\ell_1 m \mp \ell_2 n = h} \overline{\lambda_g}(m) \lambda_g(n) F(\ell_1 m, \ell_2 n)$$

Weak bounds

$$P := \mathbf{c}\ell_1\ell_2 Z(X+Y), \qquad Y \geqslant X$$

$$\Sigma_\chi^{\pm}(\ell_1,\ell_2;\mathbf{c}) \ll_{g,\varepsilon} P^\varepsilon q^{\frac{1}{2}} XY$$

$$\Sigma_\chi^{\pm}(\ell_1,\ell_2;\mathbf{c}) \ll_{g,\varepsilon} P^\varepsilon Z^{\frac{5}{4}} q^{\frac{1}{2}} X^{\frac{1}{4}} Y^{\frac{3}{2}}$$

$$\Sigma_\chi^{\pm}(\ell_1,\ell_2;\mathbf{c}) \ll_{g,\varepsilon} P^\varepsilon Z^2 \mathbf{c}^{\frac{1}{2}} X^{\frac{1}{2}} Y$$

Strong bound

$$Y \geqslant X \geqslant Z^{22+6\theta}\ell_1\ell_2\mathbf{c}^{2\theta}q^{1-2\theta-2\delta_{\mathrm{tw}}}$$

$$\Sigma_\chi^{\pm}(\ell_1,\ell_2;\mathbf{c}) \ll_{g,\varepsilon}$$

$$P^\varepsilon Z^{\frac{15+7\theta}{2(1+\theta)}}(l_1l_2)^{\frac{3+2\theta}{4(1+\theta)}}\mathbf{c}^{\frac{1+2\theta}{2(1+\theta)}}q^{\frac{1-2\theta-2\delta_{\mathrm{tw}}}{4(1+\theta)}}(Y/X)^{\frac{2}{1+\theta}}X^{\frac{1+2\theta}{4(1+\theta)}}Y$$

Circle method

$$D := \sum_{h \neq 0} \chi(h) \sum_{\ell_1 m \mp \ell_2 n = dh} \overline{\lambda_g}(m) \lambda_g(n) F(\ell_1 m, \ell_2 n) \phi(dh)$$

- pick $\ell_1 m \mp \ell_2 n = dh$ by additive characters
- approximate the circle $\mathbf{R}/\mathbf{Z}$ with characteristic functions of overlapping intervals centered at well-chosen rationals
- estimate the error by Jutila's lemma
- apply Voronoi summation to the main contribution

$$\tilde{D} = \sum_{\pm,\pm} \varepsilon_g^{\pm} \varepsilon_g^{\pm} \tilde{D}^{\pm,\pm}$$

$$\tilde{D}^{\pm,\pm} := \frac{1}{L} \sum_{m,n} \overline{\lambda_g}(m) \lambda_g(n) \tilde{D}^{\pm,\pm}(m,n)$$

$$\tilde{D}^{\pm,\pm}(m,n) := \sum_h \chi(h) \sum_{c \equiv 0\ (D\ell_1\ell_2)} \frac{S(dh, \mp\ell_1 m \pm \ell_2 n; c)}{c} \mathcal{E}^{\pm,\pm}(m,n,h;c)$$

$$D - \tilde{D} \ll_{g,\varepsilon} P^{\varepsilon} Z^2 (\ell_1\ell_2)^{1/2} \mathbf{c}^{1/2} \frac{X^{1/2} Y^{3/2}}{C}$$

- preliminary decomposition according to the sign of $\ell_1 m \pm \ell_2 n$

- separate h and $\mathbf{c}$ variables by Mellin transformation

- apply Kuznetsov's formula backwards to get a spectral decomposition

- the hth Fourier coefficients of eigenfunctions $u(z)$ in the spectrum sum up against $\frac{\chi(h)}{h^s}$ to twisted L-values $L(s, \chi \otimes u)$ for which a subconvex bound $q^{\frac{1}{2}-\delta_{\mathrm{tw}}+\varepsilon}$ is known

- We get: $\tilde{D} \ll_\varepsilon P^\varepsilon Z^{13+3\theta}(\ell_1\ell_2)C^{1+2\theta}(Y/X)^4 Y^{\frac{1}{2}-\theta}\mathbf{c}^{\frac{1}{2}+\theta}q^{\frac{1}{2}-\delta_{\mathrm{tw}}-\theta}$