The sup-norm problem for GL(2) over number fields

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Motivation for the sup-norm problem

1. Subconvexity for automorphic L-functions (periods and norms)

2. Chaos (classical and quantum)

Arithmetic quantum chaos

Consider a freely moving particle on a compact manifold M with normalized volume form $d\mathrm{vol}(m)$ and negative sectional curvature.

Classical mechanics

The particle corresponds to an orbit of the geodesic flow on the unit tangent bundle SM. The geodesic flow is ergodic, but not uniquely ergodic (there are infinitely many closed geodesics).

Quantum mechanics

The particle corresponds to an L^2-normalized linear combination of the stationary waves $\phi(m)e^{-it\sqrt{\lambda}}$, where $\triangle\phi = \lambda\phi$ and $\|\phi\|_2 = 1$.

The probability measures $|\phi(m)|^2 d\mathrm{vol}(m)$ converge weakly to $d\mathrm{vol}(m)$ along a density one subsequence of any Δ-eigenbasis $\{\phi\}$.

QUE predicts that $d\mathrm{vol}(m)$ is the only weak limit. This has been confirmed for arithmetic hyperbolic surfaces and Hecke eigenforms, and in this case GRH implies an optimal rate of convergence.

The sup-norm problem for arithmetic manifolds

Theorem (Sarnak 1994)

Let M be a compact locally symmetric space. If $\phi : M \to \mathbb{C}$ is an L^2-normalized eigenfunction of all the invariant differential operators on M, then

$$\|\phi\|_\infty \ll_M \lambda^{(\dim M - \operatorname{rank} M)/4}.$$

If M is an n-fold covering of a fixed locally symmetric space, then the implied constant is $\ll n^{1/2}$.

Problem

Assume that M is arithmetic and ϕ is a Hecke eigenform.

- *Estimate $\|\phi\|_\infty$ in terms of λ and n.*
- *Examine what happens when M is not compact.*

Results for the sup-norm problem on arithmetic manifolds

group	eigenvalue aspect	level aspect
$GL_2(\mathbb{R})$	Iwaniec–Sarnak 95	Abbes–Ullmo 95, Michel–Ullmo 98
	Rudnick 05, Xia 07	Jorgenson–Kramer 04
	Friedman–Jorgenson–Kramer 14^+	Blomer–Holowinsky 10
	Das–Sengupta 15, Steiner 16	Templier 10, Helfgott–Ricotta 11
	Templier 15	Harcos–Templier 13, Saha 15^+
	Sarnak 04, Milićević 10	Lau 10, Templier 14, Saha 15
	4 more papers since 2014	5 more papers since 2014
$GL_2(\mathbb{C})$	Koyama 95	
	Blomer–Harcos–Milićević 16	Blomer–Harcos–Milićević 16
	Rudnick–Sarnak 94, Milićević 11	
$SO_n(\mathbb{R})$	VanderKam 97, Blomer–Michel 13	Blomer–Michel 13
$Sp_4(\mathbb{R})$	Blomer–Pohl 16	
$GL_n(\mathbb{R})$	Holowinsky–Ricotta–Royer 14^+	
	Blomer–Maga 16, Marshall 14^+	
	Brumley–Templier 14^+	

Results for GL_2 and square-free level (1 of 3)

Setup

- *F is a number field with adele ring $\mathbb{A}$ and ring of integers $\mathfrak{o}$*
- *$\mathfrak{n} \subseteq \mathfrak{o}$ is a square-free ideal of norm $|\mathfrak{n}| \stackrel{\mathrm{df}}{=} [\mathfrak{o} : \mathfrak{n}]$*
- *ϕ is an L^2-normalized Hecke–Maaß cuspidal newform on $\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})$ of level $\mathfrak{n}$ and trivial central character*

Trivial bound (crude version)

Let us regard ϕ as a function on the congruence manifold

$$M := \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}) / \mathrm{Z}(F_\infty) K_0(\mathfrak{n})$$

whose connected components are left quotients of $(\mathcal{H}^2)^r \times (\mathcal{H}^3)^s$ by $\Gamma_0(\mathfrak{n})$ and related level $\mathfrak{n}$ subgroups (one for each ideal class). Sarnak's bound reads (pretending that M is compact and $n = |\mathfrak{n}|$)

$$\|\phi\|_\infty \ll \lambda^{(\dim M - \mathrm{rank}\, M)/4} |\mathfrak{n}|^{1/2} = \lambda^{[F:\mathbb{Q}]/4} |\mathfrak{n}|^{1/2}.$$

Results for GL_2 and square-free level (2 of 3)

Trivial bound (refined version)

Consider the tuple $\lambda := (\lambda_1, \dots, \lambda_r, \lambda_{r+1}, \dots, \lambda_{r+s})$ of Laplace eigenvalues at the r real places and the s complex places. Write

$$|\lambda|_\infty \stackrel{\mathrm{df}}{=} \prod_{j=1}^{r} \lambda_j \prod_{j=r+1}^{r+s} \lambda_j^2.$$

Then for ϕ a Hecke–Maaß cusp form as above, we (ought to) have

$$\|\phi\|_\infty \ll |\lambda|_\infty^{1/4} |\mathfrak{n}|^{1/2}.$$

Theorem (Templier 2012, Blomer–Harcos–Maga–Milićević 2016)

- $\|\phi\|_\infty \ll_\varepsilon |\lambda|_\infty^{5/24+\varepsilon} |\mathfrak{n}|^{1/3+\varepsilon}$ *for $F = \mathbb{Q}$*
- $\|\phi\|_\infty \ll_\varepsilon |\lambda|_\infty^{5/24+\varepsilon} |\mathfrak{n}|^{1/3+\varepsilon}$ *for F totally real*
- $\|\phi\|_\infty \ll_\varepsilon |\lambda|_\infty^{5/24+\varepsilon} |\mathfrak{n}|^{5/12+\varepsilon}$ *for F a CM-field*

Results for GL_2 and square-free level (3 of 3)

Theorem (Blomer–Harcos–Maga–Milićević 2016)

Decompose $|\lambda|_\infty$ *as* $|\lambda|_{\mathbb{R}} |\lambda|_{\mathbb{C}}$, *where*

$$|\lambda|_{\mathbb{R}} \stackrel{\mathrm{df}}{=} \prod_{j=1}^{r} \lambda_j \quad \textit{and} \quad |\lambda|_{\mathbb{C}} \stackrel{\mathrm{df}}{=} \prod_{j=r+1}^{r+s} \lambda_j^2.$$

Then for ϕ *a Hecke–Maaß cusp form as above, we have*

$$\|\phi\|_\infty \ll_\varepsilon |\lambda|_\infty^{5/24+\varepsilon} |\mathfrak{n}|^{1/3+\varepsilon} + |\lambda|_{\mathbb{R}}^{1/8+\varepsilon} |\lambda|_{\mathbb{C}}^{1/4+\varepsilon} |\mathfrak{n}|^{1/4+\varepsilon}.$$

Theorem (Blomer–Harcos–Maga–Milićević 2016)

Assume that F *is not totally real, and denote by* K *its maximal totally real subfield. Then for* ϕ *a Hecke–Maaß cusp form as above, we have*

$$\|\phi\|_\infty \ll_\varepsilon \left(|\lambda|_\infty^{1/2} |\mathfrak{n}|\right)^{\frac{1}{2} - \frac{1}{8[F:K]-4} + \varepsilon}.$$

Skeleton of the proof

As the level $\mathfrak{n}$ is square-free, the supremum of $|\phi(g)|$ is attained at a special matrix $g = \left(\begin{smallmatrix} y & x \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \theta & \\ & 1 \end{smallmatrix}\right)$, where $x \in F_\infty$, $y \in F_\infty^\times$, and $\theta \in \mathbb{A}_{\text{fin}}^\times$ lies in a fixed finite set of ideal class representatives.

We maximize $|y|_\infty$, which is partly motivated by the *Fourier bound*

$$|\phi(g)| \ll_\varepsilon \left(|\lambda|_\infty^{1/12} + |\lambda|_\infty^{1/4}\,|y|_\infty^{-1/2}\right)^{1+\varepsilon}|\mathfrak{n}|^\varepsilon.$$

If this bound is insufficient, we estimate $|\phi(g)|$ in terms of certain matrix counts by an *amplified pre-trace inequality*.

The counting is facilitated by the observation that the lattice

$$\mathfrak{o}P + \mathfrak{o} \subset \prod_{v \text{ real}} \mathbb{C} \prod_{v \text{ complex}} \mathbb{H},$$

$$P \stackrel{\text{df}}{=} \prod_{v \text{ real}} \{x_v + y_v i\} \times \prod_{v \text{ complex}} \{x_v + y_v j\},$$

has favorable diophantine properties.

Ideas for matrix counting (1 of 2)

We can derive an amplified pre-trace inequality from a suitable positive integral operator acting on $L^2(M)$ that fixes ϕ. The operator comes from an element of the underlying Hecke algebra. The inequality follows by comparing the kernels of this operator and its restriction to the invariant subspace $\mathbb{C}\phi$.

It remains to bound the number of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(F)$ such that

- γP is close to P on $(\mathcal{H}^2)^r \times (\mathcal{H}^3)^s$
- $a, d \in \mathfrak{o}$, $b \in \theta\mathfrak{o}$, $c \in \theta^{-1}\mathfrak{n}$
- $\det\gamma \in \mathfrak{o}$ is arithmetically controlled

Closeness is given in terms of parameters $0 < \delta_v \leqslant 4$ at each place $v \mid \infty$, and we seek good bounds in terms of $|\delta|_{\mathbb{R}}$ and $|\delta|_{\mathbb{C}}$. In the $|\delta|_{\mathbb{R}}$-aspect we can get good bounds by associating the lattice point

$$\gamma \mapsto (a-d)P + b \in \mathfrak{o}P + \mathfrak{o}.$$

For the $|\delta|_{\mathbb{C}}$-aspect we need additional arithmetic ideas.

Ideas for matrix counting (2 of 2)

For this last side I assume that F is not totally real and $|\delta|_{\mathbb{C}}$ is very small. I will focus on the field element $\xi \stackrel{\mathrm{df}}{=} \mathrm{tr}(\gamma)^2/\det(\gamma) \in F$.

As $|\delta|_{\mathbb{C}}$ is very small, ξ is close to being totally real. As F is not totally real, we can show that $F = K(\xi)$ cannot hold, so ξ lies in a *proper subfield* of F. However, the denominator of $\xi \in F$ is arithmetically controlled, so we infer that $\xi \in \mathfrak{o}$ is an integer. If $\xi = 4$, then γ is parabolic, for which special methods are available. In general, we only know that ξ is bounded, so we employ a trick.

We choose an auxiliary ideal $\mathfrak{q} \subseteq \mathfrak{o}$ and shrink $K_0(\mathfrak{n})$ by imposing additional congruence conditions mod $\mathfrak{q}$. Applying an amplified pre-trace inequality for the L^2-space of the covering manifold, we can ensure that the our matrices $\gamma \in \mathrm{GL}_2(F)$ are *locally parabolic* mod $\mathfrak{q}$. As a result, $\xi \in 4 + \mathfrak{q}$, which forces $\xi = 4$ when $\mathfrak{q}$ is large. To keep the determinants under control, it helps if the only units in $\mathfrak{o}$ that are quadratic residues mod $\mathfrak{q}$ are the squared units.