On the sup-norm of Maass cusp forms of large level

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7 September 2012
Synergies And Vistas in Analytic Number Theory
Mathematical Institute, University of Oxford

Overview

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The problem

Congruence subgroup

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \pmod{N} \right\}$$

Problem

Let f be a Hecke–Maass cuspidal newform on $\Gamma_0(N)\backslash\mathcal{H}$. Normalize f so that it has L^2-norm 1 with respect to $dxdy/y^2$. Estimate $\|f\|_\infty$ in terms of the Laplacian eigenvalue λ and the level N.

- Easy bounds are $\|f\|_\infty \ll_N \lambda^{1/4}$ (Seeger–Sogge 1989) and $\|f\|_\infty \ll_{\lambda,\varepsilon} N^\varepsilon$ (Abbes–Ullmo 1995).
- Better bounds rely on extra symmetries of $\Gamma_0(N)\backslash\mathcal{H}$, namely the properties of Hecke operators and Atkin–Lehner operators.
- Optimal bounds would be $\|f\|_\infty \ll_{N,\varepsilon} \lambda^{1/12+\varepsilon}$ and $\|f\|_\infty \ll_{\lambda,\varepsilon} N^{-1/4+\varepsilon}$ (cf. Templier 2012)? Not so clear.

Connections and applications

- Quantum Unique Ergodicity
- Behavior of L^p-norms of cusp forms
- Subconvex bounds for L-functions
- Bounds for exponential sums associated with cusp forms
- Bounds for shifted convolution sums of Hecke eigenvalues

Evolution of results (1 of 2)

Assume that N is square-free. Then the Atkin–Lehner operators permute the cusps of $\Gamma_0(N)\backslash\mathcal{H}$ transitively.

Theorem (Iwaniec–Sarnak 1995)

$$\|f\|_\infty \ll_{N,\varepsilon} \lambda^{5/24+\varepsilon}$$

Theorem (Blomer–Holowinsky 2010)

$$\|f\|_\infty \ll_{\lambda,\varepsilon} N^{-25/914+\varepsilon}$$

Theorem (Templier 2010)

$$\|f\|_\infty \ll_{\lambda,\varepsilon} N^{-1/22+\varepsilon}$$

Evolution of results (2 of 2)

Assume that N is square-free. Then the Atkin–Lehner operators permute the cusps of $\Gamma_0(N)\backslash\mathcal{H}$ transitively.

Theorem (Helfgott–Ricotta 2011)

$$\|f\|_\infty \ll_{\lambda,\varepsilon} N^{-1/20+\varepsilon}$$

Theorem (Harcos–Templier 2011)

$$\|f\|_\infty \ll_{\lambda,\varepsilon} N^{-1/12+\varepsilon}$$

Theorem (Harcos–Templier 2012)

$$\|f\|_\infty \ll_{\lambda,\varepsilon} N^{-1/6+\varepsilon}$$

Overview of the proof (1 of 2)

Original strategy (Iwaniec–Sarnak, Blomer–Holowinsky, Templier):

1. Pick any $z \in \mathcal{H}$ where you want to estimate $|f(z)|$.
2. Apply an Atkin–Lehner operator on z to ensure that z is not too far from the cusp ∞.
3. Use the amplification method and some trace formula to reduce the problem to a counting problem depending on z.
4. Do the counting based on the diophantine properties of z.

Improved steps in strategy (Harcos–Templier):

2. Apply an Atkin–Lehner operator on z to ensure that z is not too close to any cusp but ∞.
4. Observe that z has good diophantine properties automatically, allowing a more efficient counting.

Overview of the proof (2 of 2)

- $(f_j)_{j\geqslant 0}$ an orthonormal basis of Hecke–Maass eigenforms on $\Gamma_0(N)\backslash\mathcal{H}$ with Laplacian eigenvalues $\frac{1}{4}+r_j^2\geqslant 0$
- $h:\mathbb{R}\cup[-\frac{i}{2},\frac{i}{2}]\to(0,\infty)$ a fixed nice even function
- $a_j\geqslant 0$ is a suitable arithmetic weight for each f_j (amplifier)

We can assume that f is one of the f_j's, then by positivity

$$h(r_f)a_f\,|f(z)|^2\leqslant\sum_{j\geqslant 0}h(r_j)a_j\,|f_j(z)|^2+\text{cts}$$

From here we aim to arrive at the conclusion

$$\Lambda^{2-\varepsilon}\,|f(z)|^2\ll_{r_f,\varepsilon}\Lambda,$$

where Λ (the amplifier length) is not too small.

$$\Lambda:=N^{1/3-\varepsilon}\quad\Longrightarrow\quad f(z)\ll_{\lambda,\varepsilon}N^{-1/6+\varepsilon}.$$

Atkin–Lehner operators (1 of 3)

Definition

Atkin–Lehner operators are matrices of the form

$$W_M = \frac{1}{\sqrt{M}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}), \qquad M \mid N,$$

where $a, b, c, d \in \mathbb{Z}$ are integers satisfying

$$ad - bc = M, \qquad a \equiv 0\,(M), \qquad d \equiv 0\,(M), \qquad c \equiv 0\,(N).$$

Lemma (standard)

Let N be square-free.

1. *The W_M's form a left and right $\Gamma_0(N)$ coset for each $M \mid N$.*
2. $W_M W_{M'} = W_{M''}$ *with* $M'' := \frac{MM'}{(M,M')^2}$.
3. *Atkin–Lehner operators form a group $A_0(N)$ containing $\Gamma_0(N)$ as a normal subgroup such that $A_0(N)/\Gamma_0(N) \cong (\mathbb{Z}/2\mathbb{Z})^{\omega(N)}$.*

Atkin–Lehner operators (2 of 3)

$A_0(N)$ acts on $f(z)$ by eigenvalues ± 1, hence we can restrict z to the following fundamental domain for $A_0(N)$.

Ford polygon

$$\mathcal{F}(N) := \{z \in \mathcal{H} \mid 0 \leqslant \operatorname{Re} z \leqslant 1,\ \operatorname{Im} z \geqslant \operatorname{Im} \delta z \text{ for all } \delta \in A_0(N)\}$$

Key Lemma

Let N be square-free. For any $z \in \mathcal{F}(N)$ the associated lattice $\langle 1, z \rangle$ satisfies the following properties.

1. *The minimal distance is at least $N^{-1/2}$.*
2. *The covolume is $y = \operatorname{Im} z \gg N^{-1}$.*
3. *In any disc of radius R the number of lattice points is*

$$\ll 1 + RN^{1/2} + R^2 y^{-1}.$$

Atkin–Lehner operators (3 of 3)

Proof of the Key Lemma

1. *It suffices to show* $|cz + d| \geqslant N^{-1/2}$ *for any coprime* $c, d \in \mathbb{Z}$. *We claim that there is a unique divisor* $M \mid N$ *such that* $W_M = \frac{1}{\sqrt{M}} \left(\begin{smallmatrix} Ma & b \\ Mc & Md \end{smallmatrix} \right)$ *is an Atkin–Lehner operator for suitable* $a, b \in \mathbb{Z}$. *We need* $N \mid Mc$ *and* $Mad - bc = 1$. *The second condition can be fulfilled iff* $(Md, c) = 1$ *i.e.* (M, c)*=1. Hence* $M := N/(N, c)$ *is the unique divisor* $M \mid N$ *that works. Now*

$$\operatorname{Im} z \geqslant \operatorname{Im} W_M z = \frac{\operatorname{Im} z}{M\,|cz+d|^2} \implies |cz+d|^2 \geqslant \frac{1}{M} \geqslant \frac{1}{N}.$$

2. *The covolume is essentially the product of the two successive minima. Hence* $y = \operatorname{Im} z \gg N^{-1/2} N^{-1/2} = N^{-1}$.
3. *Consider the lattice points in a disc of radius* R. *If the points are collinear, then their number is* $\ll 1 + RN^{1/2}$. *Otherwise their number is* $\ll R^2 y^{-1}$ *by the usual Gauss argument.*

Amplification and the pretrace formula (1 of 2)

Amplifier

$$a_j := \Big(\sum_p x(p)\lambda_j(p)\Big)^2 + \Big(\sum_p x(p^2)\lambda_j(p^2)\Big)^2$$

- *sums run through the primes* $\Lambda < p < 2\Lambda$ *not dividing* N
- $\lambda_j(n)$ *is the* n*-th Hecke eigenvalue of* f_j
- $x(n)$ *abbreviates* $\operatorname{sgn}(\lambda_f(n))$

$$\lambda_f(p)^2 - \lambda_f(p^2) = 1 \quad \Longrightarrow \quad |\lambda_f(p)| + \big|\lambda_f(p^2)\big| > 1/2$$

$$\begin{aligned} a_f &= \Big(\sum_p |\lambda_f(p)|\Big)^2 + \Big(\sum_p \big|\lambda_f(p^2)\big|\Big)^2 \\ &\geqslant \frac{1}{2}\Big(\sum_p |\lambda_f(p)| + \big|\lambda_f(p^2)\big|\Big)^2 \gg_\varepsilon \Lambda^{2-\varepsilon}. \end{aligned}$$

Amplification and the pretrace formula (2 of 2)

$$\begin{aligned}\Lambda^{2-\varepsilon}\left|f(z)\right|^2 &\ll_{r_f,\varepsilon} \sum_{j\geqslant 0} h(r_j)a_j\left|f_j(z)\right|^2 + \text{cts}\\ &= \sum_{l\geqslant 1} y(l)\left(\sum_{j\geqslant 0} h(r_j)\lambda_j(l)\left|f_j(z)\right|^2 + \text{cts}\right)\\ &= \sum_{l\geqslant 1}\frac{y(l)}{\sqrt{l}} \sum_{\substack{(a,b,c,d)\in\mathbb{Z}^4\\ ad-bc=l\\ c\equiv 0\,(N)}} k\left(\frac{az+b}{cz+d}, z\right)\end{aligned}$$

$$\ll \Lambda\ M(z,1,N) + \frac{1}{\Lambda}\sum_{p_1,p_2} M(z,p_1p_2,N) + \frac{1}{\Lambda^2}\sum_{p_1,p_2} M(z,p_1^2p_2^2,N)$$

where $\Lambda < p_1, p_2 < 2\Lambda$ are primes, and $M(z,l,N)$ denotes the number of lattice points $(a,b,c,d)\in\mathbb{Z}^4$ satisfying

$$ad-bc=l,\quad c\equiv 0\,(N),\quad \left|-cz^2+(a-d)z+b\right| \leqslant N^{\varepsilon} l^{1/2} y.$$

Counting integral matrices (1 of 4)

We estimate the various sums of $M(z, l, N)$'s via

$$\left|-cz^2 + (a-d)z + b\right| \leqslant N^{\varepsilon} l^{1/2} y.$$

We treat separately the three ranges for $l = ad - bc$:
$L = 1$ for $l = 1$, $L = \Lambda^2$ for $l = p_1 p_2$, $L = \Lambda^4$ for $l = p_1^2 p_2^2$.

If $c = 0$, then $ad = l$, and for any pair (a, d) the number of choices for b is $\ll 1 + N^{\varepsilon} L^{1/2} y$.

Hence the total contribution of $M_{c=0}(z, l, N)$ is

$$\ll \Lambda(1+y) + \frac{1}{\Lambda}\Lambda^2(1+\Lambda y) + \frac{1}{\Lambda^2}\Lambda^2(1+\Lambda^2 y) \ll \Lambda + \Lambda^2 y,$$

apart from factors of N^{ε}.

Counting integral matrices (2 of 4)

From now on we assume $c \neq 0$. We prove first that

$$\max(|cz+d|, |cz-a|) \ll N^{\varepsilon} L^{1/2}.$$

This implies that

$$\#c \ll N^{\varepsilon} \frac{L^{1/2}}{Ny} \qquad \text{and} \qquad a+d \ll N^{\varepsilon} L^{1/2}.$$

We proceed in two steps, both starting from

$$\left|-cz^2 + (a-d)z + b\right| \leqslant N^{\varepsilon} l^{1/2} y.$$

Multiplying by c, $|(cz+d)(cz-a)+l| \leqslant N^{\varepsilon} l^{1/2} cy$, hence

$$\min(|cz+d|, |cz-a|) \leqslant |(cz+d)(cz-a)|^{1/2} \ll N^{\varepsilon} L^{1/2}.$$

Taking imaginary part, $|2cx+d-a| \leqslant N^{\varepsilon} l^{1/2}$, hence

$$\big||cz+d| - |cz-a|\big| \leqslant |cz+d+\overline{cz-a}| \ll N^{\varepsilon} L^{1/2}.$$

Counting integral matrices (3 of 4)

We are still using

$$\left|-cz^2 + (a-d)z + b\right| \leqslant N^{\varepsilon} l^{1/2} y.$$

For each c, the possible pairs $(a-d, b)$ correspond to lattice points from $\langle 1, z \rangle$ in a disk of radius $R \ll N^{\varepsilon} L^{1/2} y$. Hence for each c the number of choices for $(a-d, b)$ is

$$\ll N^{\varepsilon}(1 + N^{1/2} L^{1/2} y + Ly).$$

By the bounds on c and $a+d$ we see immediately that

$$\sum_{l \asymp L} M_{c \neq 0}(z, l, N) \ll N^{\varepsilon} \frac{L^{1/2}}{Ny} L^{1/2}(1 + N^{1/2} L^{1/2} y + Ly).$$

Note that here $L = 1$ or $L = \Lambda^2$ or $L = \Lambda^4$.

Counting integral matrices (4 of 4)

In the range $L = \Lambda^4$ we can do better by noting that $l = p_1^2 p_2^2$ is a square and the triple $(c, a-d, b)$ determines

$$(a+d)^2 - 4l = (a-d)^2 + 4bc.$$

Under the assumption $l < N^{-\varepsilon} y^{-2}$ we can show that the right hand side is a nonzero integer $\ll N^{\varepsilon} L$, and we observe that $a+d$ is the mean of the divisor pair $a + d \pm 2\sqrt{l}$. Hence for each triple $(c, a-d, b)$ the number of choices for $a+d$ is $\ll N^{\varepsilon}$. This furnishes the improved bound

$$\sum_{l \asymp L} M_{c \neq 0}(z, l, N) \ll N^{\varepsilon} \frac{L^{1/2}}{Ny} (1 + N^{1/2} L^{1/2} y + Ly)$$

in the range $L = \Lambda^4$, at least when $16\Lambda^4 < N^{-\varepsilon} y^{-2}$.

The endgame

The total contribution of $M_{c\neq 0}(z,l,N)$ is

$$\ll \Lambda\frac{1}{Ny}(1+N^{1/2}y+y)+\frac{1}{\Lambda}\frac{\Lambda}{Ny}\Lambda(1+N^{1/2}\Lambda y+\Lambda^2 y)$$
$$+\frac{1}{\Lambda^2}\frac{\Lambda^2}{Ny}(1+N^{1/2}\Lambda^2 y+\Lambda^4 y)\ll\frac{\Lambda}{Ny}+\frac{\Lambda^2}{N^{1/2}}+\frac{\Lambda^4}{N},$$

apart from factors of N^{ε}. Collecting all terms,

$$\Lambda^{2-\varepsilon}\left|f(z)\right|^2\ll_{\lambda,\varepsilon}\Lambda+\Lambda^2 y+\frac{\Lambda}{Ny}+\frac{\Lambda^2}{N^{1/2}}+\frac{\Lambda^4}{N}.$$

For $N^{-1}\ll y\leqslant N^{-2/3}$ and $\Lambda:=N^{1/3-\varepsilon}$ the condition $16\Lambda^4<N^{-\varepsilon}y^{-2}$ is satisfied and we obtain the desired bound:

$$\Lambda^{2-\varepsilon}\left|f(z)\right|^2\ll_{\lambda,\varepsilon}\Lambda\quad\Longrightarrow\quad f(z)\ll_{\lambda,\varepsilon}N^{-1/6+\varepsilon}.$$

For $y>N^{-2/3}$ we use the rapid decay of the Fourier expansion:

$$f(z)\ll_{\lambda,\varepsilon}N^{\varepsilon}(Ny)^{-1/2}\quad\Longrightarrow\quad f(z)\ll_{\lambda,\varepsilon}N^{-1/6+\varepsilon}.$$

Happy Birthday!