F: number field of degree d

π: irreducible cuspidal automorphic representation of GL_m over F with unitary central character and contragradient representation $\tilde{\pi}$

$$\pi = \otimes_v \pi_v$$

$$\Lambda(s,\pi) = \prod_v L(s,\pi_v)$$

$$N^{\frac{s}{2}}\Lambda(s,\pi) = \kappa N^{\frac{1-s}{2}}\Lambda(1-s,\tilde{\pi})$$

N: conductor (a positive integer)

κ: root number (of modulus 1)

$$L(s,\pi_\infty) = \prod_{v|\infty} L(s,\pi_v) = \prod_{j=1}^{md} \pi^{\frac{\mu_j - s}{2}} \Gamma\left(\frac{s-\mu_j}{2}\right)$$

$$L(s,\pi_p) = \prod_{v|p} L(s,\pi_v) = \prod_{j=1}^{md} \frac{1}{1-\alpha_j(p)p^{-s}}$$

Theorem (Luo–Rudnick–Sarnak).

$$\sup\{\Re\mu_j, \Re\log_p \alpha_j(p)\} \le \frac{1}{2} - \frac{1}{m^2+1}.$$

$$L(s,\pi) = \prod_{p<\infty} L(s,\pi_p) = \sum_{n=1}^{\infty} \frac{\lambda_\pi(n)}{n^s}, \quad \Re s > \frac{3}{2}.$$

$$C(s,\pi) = N \prod_{j=1}^{md} \frac{|s-\mu_j|}{2\pi}$$

Theorem (Molteni).

$$\sum_{n\le x} |\lambda_\pi(n)| \ll_{\epsilon,m,d} x^{1+\epsilon} C\left(\tfrac{1}{2},\pi\right)^{\epsilon}.$$

Convexity Bound. *For any* $0<\sigma<1$,

$$L(\sigma+it,\pi) \ll_{\sigma,\epsilon,m,d} C\left(\tfrac{1}{2}+it,\pi\right)^{(1-\sigma)/2+\epsilon}.$$

Lindelöf Hypothesis. *For any* $0<\sigma<1$,

$$L(\sigma+it,\pi) \ll_{\sigma,\epsilon,m,d} C\left(\tfrac{1}{2}+it,\pi\right)^{\max(0,1-2\sigma)/2+\epsilon}.$$

$$S(X,\pi) = \sum_{n=1}^{\infty} \lambda_\pi(n) w\left(\frac{n}{X}\right)$$

$$S(X,\pi) \ll_{\epsilon,w,m,d} C\left(\tfrac{1}{2},\pi\right)^{1/4-\delta+\epsilon} \sqrt{X}$$

$$\frac{1}{|\mathcal{F}|} \sum_{\rho\in\mathcal{F}} |a_\rho|^2 |S(X,\rho)|^2 \ll_{\epsilon,w,m,d} C^\epsilon X$$

$$|\mathcal{F}| \ll C^{1/2+\epsilon}, \quad \pi \in \mathcal{F}, \quad |a_\pi| \gg C^\delta$$

$$D_f(a,b;h) = \sum_{am\pm bn=h} \lambda_\pi(m)\bar{\lambda}_\pi(n) f(am,bn)$$

Theorem 1. *There is a smooth function* $f : (0, \infty) \to \mathbb{C}$ *and a complex number* λ *of modulus 1 depending only on the Archimedean parameters* μ_j $(j = 1, \ldots, md)$ *such that*

$$L\left(\frac{1}{2}, \pi\right) = \sum_{n=1}^{\infty} \frac{\lambda_\pi(n)}{\sqrt{n}} f\left(\frac{n}{\sqrt{C}}\right) + \kappa\lambda \sum_{n=1}^{\infty} \frac{\bar{\lambda}_\pi(n)}{\sqrt{n}} \bar{f}\left(\frac{n}{\sqrt{C}}\right).$$

The function f *and its partial derivatives* $f^{(k)}$ $(k = 1, 2, \ldots.)$ *satisfy the following uniform growth estimates at* 0 *and infinity:*

$$f(x) = \begin{cases} 1 + O_\sigma(x^\sigma), & 0 < \sigma < \frac{1}{m^2+1}; \\ O_\sigma(x^{-\sigma}), & \sigma > 0; \end{cases}$$

$$f^{(k)}(x) = O_{\sigma,k}(x^{-\sigma}), \quad \sigma > k - \frac{1}{m^2+1}.$$

The implied constants depend only on σ, k, m *and* d.

$$\eta = \min_{j=1,\ldots,md} \left| \frac{1}{2} - \mu_j \right|.$$

Theorem 2. *Let $g : (0, \infty) \to \mathbb{R}$ be a smooth function with the functional equation $g(x) + g(1/x) = 1$ and derivatives decaying faster than any negative power of x as $x \to \infty$. Then*

$$L\left(\frac{1}{2}, \pi\right) = \sum_{n=1}^{\infty} \frac{\lambda_\pi(n)}{\sqrt{n}} g\left(\frac{n}{\sqrt{C}}\right) + \kappa\lambda \sum_{n=1}^{\infty} \frac{\bar{\lambda}_\pi(n)}{\sqrt{n}} g\left(\frac{n}{\sqrt{C}}\right)$$

$$+ O_{\epsilon,g}(\eta^{-1} C^{1/4+\epsilon}),$$

where λ (of modulus 1) is as before, and the implied constant depends only on ϵ, g, m and d.

Theorem 3. *Suppose that ϕ is a primitive holomorphic or Maass cusp form of Archimedean size $|\tilde{\mu}|$, level N and arbitrary nebentypus character mod N. Let $\Re s = 1/2$ and q be an integer prime to N. If χ is a primitive Dirichlet character modulo q, then*

$$L(s, \phi \otimes \chi) \ll |s|^{1+\epsilon} N^{9/8+\epsilon} |\tilde{\mu}|^{27/20+\epsilon} q^{1/2-1/54+\epsilon},$$

where the implied constant depends only on ϵ.

Theorem 4. *Let* $\lambda_\phi(m)$ *(resp.* $\lambda_\psi(n)$*) be the normalized Fourier coefficients of a holomorphic or Maass cusp form* ϕ *(resp.* ψ*) of level* N *and arbitrary nebentypus character modulo* N*. Let* $|\tilde{\mu}|$ *(resp.* $|\tilde{\nu}|$*) denote the Archimedean size of* ϕ *(resp.* ψ*), and assume that the partial derivatives of the weight function* $f : (0, \infty) \times (0, \infty) \to \mathbb{C}$ *satisfy the estimate*

$$x^k y^l f^{(k,l)}(x, y) \ll_{k,l} \left(1 + \frac{x}{X}\right)^{-1} \left(1 + \frac{y}{Y}\right)^{-1} P^{k+l}$$

with some $P, X, Y \geq 1$ *for all* $k, l \geq 0$*. Then for* $h > 0$ *and coprime* $a > 0$ *and* $b > 0$ *we have*

$$\sum_{am \pm bn = h} \lambda_\phi(m) \lambda_\psi(n) f(am, bn) \ll$$

$$P^{11/10} N^{9/5} |\tilde{\mu}\tilde{\nu}|^{9/5+\epsilon} (ab)^{-1/10} (X + Y)^{1/10} (XY)^{2/5+\epsilon},$$

where the implied constant depends only on ϵ*.*

$$\phi(x+iy) = \sum_{n\neq 0} \rho_\phi(n) W(ny) e(nx),$$

$$W(y) = \begin{cases} e^{-2\pi y} & \text{if } \phi \text{ is holomorphic,} \\ |y|^{1/2} K_{i\mu}\big(2\pi|y|\big) & \text{if } \phi \text{ is real-analytic.} \end{cases}$$

$$\langle \phi, \phi \rangle = \int_{\Gamma_0(N)\backslash\mathcal{H}} y^k |\phi(x+iy)|^2 \frac{dx\,dy}{y^2},$$

$$\lambda_\phi(n) = \begin{cases} \left(\frac{N(k-1)!}{\langle\phi,\phi\rangle(4\pi n)^{k-1}}\right)^{1/2} \rho_\phi(n) & \text{if } \phi \text{ is holomorphic,} \\ \left(\frac{N(4\pi|n|)}{\langle\phi,\phi\rangle \cosh \pi\mu}\right)^{1/2} \rho_\phi(n) & \text{if } \phi \text{ is real-analytic.} \end{cases}$$

Lemma 1. *For any $\epsilon > 0$ there is a uniform bound*

$$\left\| y^{k/2}\phi(x+iy) \right\|_\infty \ll \langle \phi, \phi \rangle^{1/2} |\tilde{\mu}|^{3/2+\epsilon}.$$

The implied constant depends only on ϵ.

Proposition 1. *For any $\epsilon > 0$ there is a uniform bound*

$$\sum_{1 \le m \le M} \lambda_\phi(m) e(\alpha m) \ll N^{1/2} |\tilde{\mu}|^{2+\epsilon} M^{1/2+\epsilon}, \quad \alpha \in \mathbb{R},\ M > 0.$$

The implied constant depends only on ϵ.

Proposition 2 (Jutila). *Let $\mathcal{Q}$ be a nonempty set of integers $Q \le q \le 2Q$, where $Q \ge 1$. Let $Q^{-2} \le \delta \le Q^{-1}$, and for each fraction d/q (in its lowest terms) denote by $I_{d/q}(\alpha)$ the characteristic function of the interval $[d/q - \delta, d/q + \delta]$. Write L for the number of such intervals, that is,*

$$L = \sum_{q \in \mathcal{Q}} \varphi(q),$$

and put

$$\tilde{I}(\alpha) = \frac{1}{2\delta L} \sum_{q \in \mathcal{Q}} \ \sideset{}{^*}\sum_{d \pmod{q}} I_{d/q}(\alpha).$$

If $I(\alpha)$ is the characteristic function of the unit interval $[0,1]$, then

$$\int_{-\infty}^{\infty} \Big(I(\alpha) - \tilde{I}(\alpha) \Big)^2 \, d\alpha \ll \delta^{-1} L^{-2} Q^{2+\epsilon},$$

where the implied constant depends on ϵ only.

Proposition 3 (Meurman). *Let d and q be coprime integers such that $N \mid q$, and let g be a smooth, compactly supported function on $(0, \infty)$. If ϕ is a real-analytic Maass cusp form of level N, nebentypus χ and nonnegative Laplacian eigenvalue $1/4 + \mu^2$ then*

$$\chi(d) \sum_{n=1}^{\infty} \lambda_\phi(n) e_q(dn) g(n) = \sum_{\pm} \sum_{n=1}^{\infty} \lambda_\phi(\mp n) e_q\left(\pm \bar{d} n\right) g^{\pm}(n),$$

where

$$g^{-}(y) = -\frac{\pi}{q \cosh \pi\mu} \int_0^\infty g(x) \{Y_{2i\mu} + Y_{-2i\mu}\} \left(\frac{4\pi\sqrt{xy}}{q} \right) dx,$$

$$g^{+}(y) = \frac{4 \cosh \pi\mu}{q} \int_0^\infty g(x) K_{2i\mu} \left(\frac{4\pi\sqrt{xy}}{q} \right) dx.$$

Here $\bar{d}$ is a multiplicative inverse of $d \bmod q$, $e_q(x) = e(x/q) = e^{2\pi i x/q}$, and $Y_{\pm 2i\mu}$, $K_{2i\mu}$ are Bessel functions.

Proposition 4. *For any $\sigma > 0$ and $\epsilon > 0$ the following uniform estimates hold in the strip $|\Re s| \le \sigma$:*

$$e^{-\pi|\Im s|/2} Y_s(x) \ll \begin{cases} \Big(1+|\Im s|\Big)^{\sigma+\epsilon} x^{-\sigma-\epsilon}, & 0 < x \le 1+|\Im s|; \\ \Big(1+|\Im s|\Big)^{-\epsilon} x^{\epsilon}, & 1+|\Im s| < x \le 1+|s|^2; \\ x^{-1/2}, & 1+|s|^2 < x. \end{cases}$$

$$e^{\pi|\Im s|/2} K_s(x) \ll \begin{cases} \Big(1+|\Im s|\Big)^{\sigma+\epsilon} x^{-\sigma-\epsilon}, & 0 < x \le 1+\pi|\Im s|/2; \\ e^{-x+\pi|\Im s|/2} x^{-1/2}, & 1+\pi|\Im s|/2 < x. \end{cases}$$

The implied constants depend only on σ and ϵ.

$$\phi(z) = \sum_{n=1}^{\infty} \rho_\phi(n) e(nz)$$

$$\sum_{n=1}^{\infty} \frac{|\rho_\phi(n)|^2}{n^{s+k-1}} = \frac{(4\pi)^{s+k-1}}{\Gamma(s+k-1)} \int_{\Gamma\backslash\mathcal{H}} y^k |\phi(z)|^2 E(z,s) \frac{dx\,dy}{y^2}$$

$$E(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} y^s(\gamma z), \qquad \Gamma = \Gamma_0(N)$$

$$\sum_{am-bn=h} \frac{\rho_\phi(m)\bar{\rho}_\phi(n)}{(am+bn)^{s+k-1}} = \frac{(2\pi)^{s+k-1}}{\Gamma(s+k-1)} \int_{\Gamma\backslash\mathcal{H}} y^k \phi(az)\bar{\phi}(bz) P_h(z,s) \frac{dx\,dy}{y^2}$$

$$P_h(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} y^s(\gamma z) e(-hx(\gamma z)), \qquad \Gamma = \Gamma_0(Nab)$$

$$\phi(x+iy) = \sum_{n\neq 0} \rho_\phi(n) \tilde{W}_{\frac{n}{|n|}\frac{\kappa}{2}, i\mu}\Big(4\pi|n|y\Big) e(nx),$$

where

$$\tilde{W}_{\alpha,\beta}(y) = \left\{ \frac{\Gamma\left(\frac{1}{2}+\beta-\alpha\right)}{\Gamma\left(\frac{1}{2}+\beta+\alpha\right)} \right\}^{1/2} W_{\alpha,\beta}(y),$$

$$W_{\alpha,\beta}(y) = \frac{e^{y/2}}{2\pi i} \int_{(\sigma)} \frac{\Gamma(w-\beta)\Gamma(w+\beta)}{\Gamma\left(\frac{1}{2}+w-\alpha\right)} y^{\frac{1}{2}-w} \, dw, \qquad \sigma > |\Re\beta|.$$

is the (normalized) Whittaker function. The normalization is introduced in order to retain the coefficients $\rho_\phi(\pm n)$ after the Maass operators have been applied.

If k is an integer of the same parity as κ, then

$$\phi_k(x+iy) = \sum_{n\neq 0} \rho_\phi(n) \tilde{W}_{\frac{n}{|n|}\frac{k}{2}, i\mu}\Big(4\pi|n|y\Big) e(nx)$$

is a Maass form of weight k and the same Petersson norm as ϕ.

$$(2\pi h)^{s-1} \int_{\Gamma\backslash\mathcal{H}} \phi_k(az)\bar{\phi}_k(bz) P_h(z,s) \frac{dx\,dy}{y^2}$$

$$= \sum_{am-bn=h} \rho_\phi(m)\bar{\rho}_\phi(n) H_{s,k,i\mu}\left(\frac{am+bn}{h}\right),$$

$$H_{s,k,i\mu}(u) = \int_0^\infty \tilde{W}_{\frac{u+1}{|u+1|}\frac{k}{2}, i\mu}\Big(|u{+}1|y\Big) \bar{\tilde{W}}_{\frac{u-1}{|u-1|}\frac{k}{2}, i\mu}\Big(|u{-}1|y\Big) y^{s-2}\, dy, \qquad u \neq \pm 1.$$

$$H_{s,0,i\mu}(u) = M(s)u|1-u^{-2}|^{\frac{1}{2}+i\mu}G_s(u),$$

$$M(s) = \frac{2^{2s-3}\Gamma\left(\frac{s}{2}-i\mu\right)\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+i\mu\right)}{\pi\Gamma(s)},$$

$$G_s(u) = \begin{cases} u^{2i\mu}F\left(\frac{s}{2}+i\mu, \frac{1}{2}+i\mu; \frac{s}{2}+\frac{1}{2}; u^2\right), & 0 \le u < 1; \\ u^{-s}F\left(\frac{s}{2}+i\mu, \frac{1}{2}+i\mu; \frac{s}{2}+\frac{1}{2}; u^{-2}\right), & 1 < u. \end{cases}$$

$$G_s(u) = \begin{cases} u^{2i\mu} F\left(\frac{s}{2} + i\mu, \frac{1}{2} + i\mu; \frac{s}{2} + \frac{1}{2}; u^2\right), & 0 \le u < 1; \\ u^{-s} F\left(\frac{s}{2} + i\mu, \frac{1}{2} + i\mu; \frac{s}{2} + \frac{1}{2}; u^{-2}\right), & 1 < u. \end{cases}$$

Definition. Let V be an arbitrary complex valued function on the positive axis $(0, \infty)$, and $V^\bigstar(s)$ be a complex valued function on the vertical line $\sigma + i\mathbb{R}$ $(\sigma > 1)$, such that

$$\int_{(\sigma)} |s|^{3/2+\epsilon} |V^\bigstar(s)| \, ds < \infty$$

holds for some $\epsilon > 0$. Then $V^\bigstar$ is a $\bigstar$ transform of V if

$$V(u) = \frac{1}{2\pi i} \int_{(\sigma)} V^\bigstar(s) G_s(u) \, ds$$

holds for all $u > 0$, $u \neq 1$.

Lemma 2. *Let* $\sigma > 1$ *and* $0 \le u < 1$. *Then for any* $\epsilon > 0$ *the following uniform bound holds on the vertical line* $\sigma + i\mathbb{R}$*:*

$$F\left(\frac{s}{2} + i\mu, \frac{1}{2} + i\mu; \frac{s}{2} + \frac{1}{2}; u\right) \ll |s|^{1/2+\epsilon}.$$

The implied constant depends only on σ *and* ϵ*.*

Lemma 3. *Let* $\sigma > 1$ *and* $0 \le v < u < 1$. *Then for any* $\epsilon > 0$ *the following uniform bound holds on the vertical line* $\sigma + i\mathbb{R}$*:*

$$F\left(\frac{s}{2} + i\mu, \frac{1}{2} + i\mu; \frac{s}{2} + \frac{1}{2}; u\right) - F\left(\frac{s}{2} + i\mu, \frac{1}{2} + i\mu; \frac{s}{2} + \frac{1}{2}; v\right) \ll (u-v)|s|^{3/2+\epsilon}.$$

Formula 2.21.1.3 from A. P. Prudnikov, Yu. A. Brychkov, O. I. Marichev, Integrals and series, Vol. 3 (More special functions), Gordon and Breach Science Publishers, New York, 1986:

$$\frac{\Gamma(\alpha)\Gamma(a-\alpha)}{\Gamma(1-b+\alpha)\Gamma(c-\alpha)} = \frac{\Gamma(a)}{\Gamma(1-b)\Gamma(c)} \int_0^1 u^\alpha F(a,b;c;u)\frac{du}{u}$$

$$+ \frac{\Gamma(a)}{\Gamma(c-a)\Gamma(a-b+1)} \int_1^\infty u^{\alpha-a} F\left(a, a-c+1; a-b+1; \frac{1}{u}\right) \frac{du}{u}.$$

Lemma 4. *For* $0 < \Re z < \sigma$ *the Mellin transform of* $G_s(u)$ *is given by*

$$G_s^*(z) = c_{i\mu} \frac{\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}+i\mu\right)} \frac{\Gamma\left(\frac{s-z}{2}\right)}{\Gamma\left(\frac{s-z}{2}+\frac{1}{2}-i\mu\right)} \frac{\Gamma\left(\frac{z}{2}+i\mu\right)}{\Gamma\left(\frac{z}{2}+\frac{1}{2}\right)},$$

where $c_{i\mu}$ *abbreviates the constant* $\frac{1}{2}\Gamma\left(\frac{1}{2}-i\mu\right)$.

Theorem 5. *Suppose that an arbitrary function* $V : (0, \infty) \to \mathbb{C}$ *has a* $\bigstar$ *transform on the vertical line* $\sigma + i\mathbb{R}$ *(*$\sigma > 1$*). Then* V *is continuous at all points* $u \neq 1$*, the Mellin transform* $V^*(z)$ *of* V *is defined in* $0 < \Re z < 1$*, and* $\frac{\Gamma\left(\frac{z}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{z}{2}+i\mu\right)} V^*(z)$ *extends to a bounded holomorphic function in every half-plane* $\Re z < \sigma_0 < \sigma$*. Conversely, let* $V : (0, \infty) \to \mathbb{C}$ *be an arbitrary function which is continuous at all points* $u \neq 1$ *and has Mellin transform* $V^*(z)$ *defined in* $0 < \Re z < 1$*. If* $K(z) = \frac{\Gamma\left(\frac{z}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{z}{2}+i\mu\right)} V^*(z)$ *extends to a holomorphic function in some half-plane* $\Re z < \sigma_0$ *(*$\sigma_0 > 1$*) satisfying* $K(z) \ll (1 + |z|)^{-A}$ *for some* $A > 2$*, then* V *has a* $\bigstar$ *transform* $V^{\bigstar}(s)$*, which extends to a holomorphic function in* $0 < \Re s < \sigma_0$ *satisfying* $V^{\bigstar}(s) \ll_{\sigma, A} \left(1 + |s|\right)^{-A-1/2}$*.*

Corollary 1. *Let* $V : (0,\infty) \to \mathbb{C}$ *be an arbitrary function compactly supported in* $(0,1) \cup (1,\infty)$*. If* V *has a* ★ *transform, then it is identically zero.*

Corollary 2. *Let* $V : (0,\infty) \to \mathbb{C}$ *be an arbitrary function. If* $V(u/c)$ *has a* ★ *transform for every* $c > 0$*, then* V *is identically zero.*