A new zero-free region for Rankin–Selberg L-functions

(joint work with Jesse Thorner)

Gergely Harcos

Alfréd Rényi Institute of Mathematics
https://users.renyi.hu/∼gharcos/

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Nonvanishing of Dirichlet L-functions (1 of 2)

Dirichlet (1837) and Riemann (1859) observed that the logarithm of a Dirichlet L-function is supported on prime powers, hence it can be used to study the distribution of primes up to a given height in various residue classes.

One can define $\log L(s,\psi)$ as a holomorphic function in any simply connected region where $L(s,\psi) \neq 0$, but the derivative of this function is more pleasant to work with:

$$-\frac{L'(s,\psi)}{L(s,\psi)} = \sum_{m=1}^{\infty} \frac{\Lambda(m)\psi(m)}{m^s}, \qquad \mathrm{Re}(s) > 1.$$

Generalized Riemann hypothesis: $L(s,\psi) \neq 0$ for $\mathrm{Re}(s) > 1/2$.

Nonvanishing of Dirichlet L-functions (2 of 2)

Work of Hadamard (1896), de la Vallée Poussin (1896 & 1899), Gronwall (1913), Landau (1918), Titchmarsh (1930), Page (1935), Siegel (1935) led to the following zero-free region and more.

Theorem

There exists a constant $c_1 > 0$ with the following property. If ψ is a primitive Dirichlet character modulo q, then $L(s, \psi)$ has at most one zero (necessarily real and simple) in the region

$$\mathrm{Re}(s) \geqslant 1 - c_1 / \log(q(|\mathrm{Im}(s)| + 3)).$$

If the exceptional zero exists, then ψ is quadratic. In this case, for all $\varepsilon > 0$, there exists a constant $c_2 = c_2(\varepsilon) > 0$ such that

$$L(\sigma, \psi) \neq 0, \qquad \sigma \geqslant 1 - c_2 q^{-\varepsilon}.$$

In fact this theorem comes with lower bounds (as opposed to just nonvanishing) that we omit for simplicity.

Primes in arithmetic progressions

We extract the following estimate, especially because this is what we shall generalize to GL_1-twists of $\mathrm{GL}_m \times \mathrm{GL}_n$ L-functions.

Theorem (de la Vallée Poussin 1899, Siegel 1935)

For all $\varepsilon > 0$, there exists an ineffective constant $c_3 = c_3(\varepsilon) > 0$ such that for all primitive Dirichlet characters ψ modulo q,

$$|L(\sigma + it, \psi)| \geqslant c_3(q + |t|)^{-\varepsilon}, \qquad \sigma \geqslant 1 - c_3(q + |t|)^{-\varepsilon}.$$

Theorem (Walfisz 1936)

Let $A > 0$ be arbitrary. Let $q \leqslant (\log x)^A$ be a positive integer, and let $a \pmod q$ be a reduced residue class modulo q. Then

$$\sum_{\substack{m \leqslant x \\ m \equiv a \pmod q}} \Lambda(m) = \frac{x}{\varphi(q)} + O_A\left(\frac{x}{(\log x)^A}\right).$$

Standard L-functions

Notation

Let $\mathfrak{F}_n$ be the set of unitary cuspidal automorphic representations of GL_n over $\mathbb{Q}$. Let $\mathfrak{F}_n^* \subset \mathfrak{F}_n$ be the subset of representations in $\mathfrak{F}_n$ whose central character is trivial on the positive reals.

Each $\pi \in \mathfrak{F}_n$ has a standard L-function

$$L(s,\pi) = \prod_p \prod_{j=1}^{n} \frac{1}{1-\alpha_{j,\pi}(p)p^{-s}}, \qquad \mathrm{Re}(s) > 1.$$

By the results of Tamagawa (1963), Satake (1963), and Kondo–Yasuda (2010), the coefficients of the denominator (as a polynomial of p^{-s}) are Hecke eigenvalues on newforms in π.

The L-function is completed with n gamma factors, and the resulting $\Lambda(s,\pi)$ satisfies a similar functional equation as Dirichlet L-functions. (A product of n Dirichlet L-functions is the L-function of an isobaric automorphic representation of GL_n over $\mathbb{Q}$.)

Twists of automorphic representations

Example

$\mathfrak{F}_1$ is the abelian group of unitary Hecke characters acting on $\mathfrak{F}_n$ as follows. For each $\pi \in \mathfrak{F}_n$ and $\chi \in \mathfrak{F}_1$, the representation $\pi \otimes \chi \in \mathfrak{F}_n$ given by $g \mapsto \pi(g)\chi(\det g)$ for $g \in \mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$.

Example

Each $\chi \in \mathfrak{F}_1$ corresponds bijectively to a pair (t, ψ), where $t \in \mathbb{R}$ and ψ is a primitive Dirichlet character (including the trivial character). Under this correspondence, $L(s, \chi) = L(s + it, \psi)$. More generally, $L(s, \pi \otimes \chi) = L(s + it, \pi \otimes \psi)$ for all $\pi \in \mathfrak{F}_n$.

Nonvanishing of standard L-functions

The results of de la Vallée Poussin (1899) and Siegel (1935) for Dirichlet L-functions have been extended to standard automorphic L-functions. Important milestones include Jacquet–Shalika (1976), Moreno (1985), Hoffstein–Ramakrishnan (1995), Iwaniec–Kowalski (2014), Brumley (2019) and Jiang–Lü–Thorner–Wang (2021).

Theorem

There exists a constant $c_4 = c_4(n) > 0$ with the following property. If $\pi \in \mathfrak{F}_n^*$, then $L(s,\pi)$ has at most one zero (necessarily real and simple) in the region

$$\mathrm{Re}(s) \geqslant 1 - c_4/\log(C(\pi)(|\mathrm{Im}(s)|+3)).$$

If the exceptional zero exists, then π is self-dual. If π is self-dual, then for all $\varepsilon > 0$, there exists a constant $c_5 = c_5(\pi,\varepsilon)$ such that $L(\sigma, \pi\otimes\chi) \neq 0$ for all quadratic $\chi \in \mathfrak{F}_1^*$ and $\sigma \geqslant 1 - c_5 C(\chi)^{-\varepsilon}$

Rankin–Selberg L-functions

For each $(\pi, \rho) \in \mathfrak{F}_n \times \mathfrak{F}_m$, there is a Rankin–Selberg L-function

$$L(s, \pi \times \rho) = \prod_{p} \prod_{j=1}^{n} \prod_{k=1}^{m} \frac{1}{1 - \alpha_{j,k,\pi\times\rho}(p) p^{-s}}, \qquad \mathrm{Re}(s) > 1.$$

- If p is an unramified prime for π and ρ, then we can take $\alpha_{j,k,\pi\times\rho}(p) = \alpha_{j,\pi}(p)\alpha_{k,\rho}(p)$.
- The L-function is completed with nm gamma factors, and the resulting $\Lambda(s, \pi \times \rho)$ satisfies a similar functional equation as standard L-functions.
- Langlands functoriality predicts that every Rankin–Selberg L-function as above is a standard L-function. Hoffstein–Ramakrishnan (1995) used this hypothesis to prove the non-existence of Landau–Siegel zeros other than those of Dirichlet L-functions.

Nonvanishing of Rankin–Selberg L-functions

Shahidi (1981) proved that $L(s,\pi\times\rho)\neq 0$ for $\mathrm{Re}(s)\geqslant 1$. This has been strengthened in various ways by Moreno (1985), Sarnak (2004), Goldfeld–Li (2018), Humphries (2019) and Zhang (2023).

Theorem (Brumley 2006–2019, Humphries–Thorner 2022)

There exists $c_6=c_6(n,m)>0$ with the following property. If $(\pi,\rho)\in\mathfrak{F}_n^*\times\mathfrak{F}_m^*$, then $L(s,\pi\times\rho)$ has no zero in the region

$$\mathrm{Re}(s)\geqslant 1-c_6(C(\pi)C(\rho))^{-n-m}(|\mathrm{Im}(s)|+1)^{-nm}.$$

Moreover, if $\pi=\widetilde{\pi}$ or $\rho=\widetilde{\rho}$ or $\rho=\widetilde{\pi}$, then $L(s,\pi\times\rho)$ has at most one zero (necessarily real and simple) in the region

$$\mathrm{Re}(s)\geqslant 1-c_6/\log(C(\pi)C(\rho)(|\mathrm{Im}(s)|+3)).$$

If the exceptional zero exists, then $(\pi,\rho)=(\widetilde{\pi},\widetilde{\rho})$ or $\rho=\widetilde{\pi}$.

A new zero-free region

We extended the celebrated lower bound of Siegel (1935) to all GL_1-twists of general Rankin–Selberg L-functions. Special cases were established earlier by Jiang–Lü–Thorner–Wang (2021) and Humphries–Thorner (2024).

Theorem (Harcos–Thorner)

Let $(\pi,\rho)\in\mathfrak{F}_n\times\mathfrak{F}_m$. For all $\varepsilon>0$, there exists an ineffective constant $c_7=c_7(\pi,\rho,\varepsilon)>0$ such that if $\chi\in\mathfrak{F}_1$, then

$$|L(\sigma,\pi\times(\rho\otimes\chi))|\geqslant c_7 C(\chi)^{-\varepsilon},\qquad \sigma\geqslant 1-c_7 C(\chi)^{-\varepsilon}.$$

The proof relies on the group structure of $\mathfrak{F}_1$, and it utilizes an auxiliary L-function with nonnegative coefficients that extends the constructions of de la Vallée Poussin (1899) and Siegel (1935).

An analogue of the Siegel–Walfisz theorem

The new zero-free region allows us to prove an analogue of the Siegel–Walfisz theorem for Rankin–Selberg L-functions.

Notation

For $(\pi, \rho) \in \mathfrak{F}_n \times \mathfrak{F}_m$, let $\Lambda_{\pi\times\rho}(m)$ denote the m-th Dirichlet coefficient of $-L'(s, \pi \times \rho)/L(s, \pi \times \rho)$. Moreover, let

$$\mathcal{M}_{\pi\times\rho}(x) = \begin{cases} x^{1-iu}/(1-iu), & \rho = \widetilde{\pi} \otimes |\cdot|^{iu} \\ 0, & \text{otherwise} \end{cases}$$

Theorem (Harcos–Thorner)

Let $(\pi, \rho) \in \mathfrak{F}_n \times \mathfrak{F}_m$. Let $A > 0$ be arbitrary. Let $q \leqslant (\log x)^A$ be a positive integer coprime to the conductors of π and ρ, and let $a \pmod{q}$ be a reduced residue class modulo q. Then

$$\sum_{\substack{m \leqslant x \\ m \equiv a \pmod{q}}} \Lambda_{\pi\times\rho}(m) = \frac{\mathcal{M}_{\pi\times\rho}(x)}{\varphi(q)} + O_{\pi,\rho,A}\left(\frac{x}{(\log x)^A}\right).$$

Symmetric power L-functions

Our second application is based on cases of functoriality established by Gelbart–Jacquet (1978), Kim–Shahidi (2002) and Kim (2003).

Theorem (Harcos–Thorner)

Let $\pi \in \mathfrak{F}_2$ and $n \in \{1, \ldots, 8\}$. Assume that $L(s, \pi, \mathrm{Sym}^n \otimes \chi)$ has no pole in the half-plane $\mathrm{Re}(s) \geqslant 1$. There exists $c_8 = c_8(\pi, \varepsilon) \geqslant 1$ such that for all $\chi \in \mathfrak{F}_1$ and $\sigma \geqslant 1 - c_8^{-1} C(\chi)^{-\varepsilon}$,

$$c_8^{-1} C(\chi)^{-\varepsilon} \leqslant |L(\sigma, \pi, \mathrm{Sym}^n \otimes \chi)| \leqslant c_8 C(\chi)^{\varepsilon}.$$

For $n \in \{5, 6, 7, 8\}$, the idea is to use the identity

$$L(s, \pi, \mathrm{Sym}^n \otimes \chi) = \frac{L(s, \mathrm{Sym}^4(\pi) \times (\mathrm{Sym}^{n-4}(\pi) \otimes \chi))}{L(s, \mathrm{Sym}^3(\pi) \times (\mathrm{Sym}^{n-5}(\pi) \otimes \chi\omega_\pi))}.$$

Strategy of the proof

By the convexity bound for $L'(s, \pi \times (\rho \otimes \chi))$, it suffices to prove

$$|L(1, \pi \times (\rho \otimes \chi))| \gg_{\pi,\rho,\varepsilon} C(\chi)^{-\varepsilon}, \qquad \chi \in \mathfrak{F}_1.$$

We prove the above bound in three steps. In each step, we prove it for a certain subgroup of characters $G \leqslant \mathfrak{F}_1$ containing the vertical shift characters.

In each step, we treat separately the characters $\chi \in G$ for which $L(s, \pi \times (\rho \otimes \chi))$ has a pole. So let us focus on the case when $L(s, \pi \times (\rho \otimes \chi))$ has no pole. If all these entire twists by $\chi \in G$ satisfy a quasi Riemann hypothesis, then we are done.

So we can assume that there exists an exceptional $\widetilde{\chi} \in G$ such that $L(s, \pi \times (\rho \otimes \widetilde{\chi}))$ is entire and has a real zero close to 1. We fix $\widetilde{\chi} \in G$ in terms of $(\pi, \rho, G, \varepsilon)$. At this point we rename $\rho \otimes \widetilde{\chi}$ to ρ, so the assumption is really that $L(s, \pi \times \rho)$ is entire and has a real zero close to 1. This assumption is quite powerful as we shall see.

The Key Proposition

The three subgroups $G \leqslant \mathfrak{F}_1$ corresponding to the three steps are:

1. the subgroup of vertical shifts
2. the subgroup generated by vertical shifts and quadratic characters
3. the full group of unitary Hecke characters

In each of the three steps we apply the following

Key Proposition

Let $(\pi, \rho, \chi) \in \mathfrak{F}_n \times \mathfrak{F}_m \times \mathfrak{F}_1$, $\varepsilon \in (0, 1/2)$, *and* $\beta \in (1 - \varepsilon/8, 1)$. *Assume that the following L-functions are entire:*

$$L(s, \pi \times \rho), \qquad L(s, \pi \times (\rho \otimes \chi)), \qquad L(s, \pi \times (\rho \otimes \chi^2)).$$

If $L(\beta, \pi \times \rho) = 0$, *then*

$$|L(1, \pi \times (\rho \otimes \chi))| \gg_{\pi,\rho,\beta,\varepsilon} C(\chi)^{-(n+m)^2\varepsilon}.$$

The auxiliary L-function

The proof of the Key Proposition utilizes the auxiliary L-function

$$D(s) = L(s, \Pi \times \widetilde{\Pi}), \qquad \Pi = \pi \boxplus \pi \otimes \chi \boxplus \widetilde{\rho} \boxplus \widetilde{\rho} \otimes \overline{\chi}.$$

This auxiliary L-function has nonnegative Dirichlet coefficients by a result of Hoffstein–Ramakrishnan (1995), and it factors as

$$\begin{aligned}
&L(s, \pi \times \widetilde{\pi})^2 L(s, \rho \times \widetilde{\rho})^2 L(s, \pi \times (\rho \otimes \chi))^2 L(s, \widetilde{\pi} \times (\widetilde{\rho} \otimes \overline{\chi}))^2 \\
&L(s, \pi \times (\widetilde{\pi} \otimes \chi)) L(s, \rho \times (\widetilde{\rho} \otimes \chi)) L(s, \widetilde{\pi} \times \widetilde{\rho}) L(s, \pi \times (\rho \otimes \chi^2)) \\
&L(s, \pi \times (\widetilde{\pi} \otimes \overline{\chi})) L(s, \rho \times (\widetilde{\rho} \otimes \overline{\chi})) L(s, \pi \times \rho) L(s, \widetilde{\pi} \times (\widetilde{\rho} \otimes \overline{\chi}^2)).
\end{aligned}$$

This auxiliary L-function can have three distinct poles (with various multiplicities), but what makes the proof work is the following. If s_0 is a pole of $D(s)$ of multiplicity k, then there are at least k factors above whose size at $s = s_0$ equals $|L(1, \pi \times (\rho \otimes \chi))|$.

Back to de la Vallée Poussin (1899) and Siegel (1935)

Let us look at the auxiliary L-function $D(s) = L(s, \Pi \times \widetilde{\Pi})$ again:

$$L(s, \pi \times \widetilde{\pi})^2 L(s, \rho \times \widetilde{\rho})^2 L(s, \pi \times (\rho \otimes \chi))^2 L(s, \widetilde{\pi} \times (\widetilde{\rho} \otimes \overline{\chi}))^2$$
$$L(s, \pi \times (\widetilde{\pi} \otimes \chi)) L(s, \rho \times (\widetilde{\rho} \otimes \chi)) L(s, \widetilde{\pi} \times \widetilde{\rho}) L(s, \pi \times (\rho \otimes \chi^2))$$
$$L(s, \pi \times (\widetilde{\pi} \otimes \overline{\chi})) L(s, \rho \times (\widetilde{\rho} \otimes \overline{\chi})) L(s, \pi \times \rho) L(s, \widetilde{\pi} \times (\widetilde{\rho} \otimes \overline{\chi}^2)).$$

In the special case $\pi = \rho = \mathbb{1}$, this becomes

$$D(s) = \zeta(s)^6 L(s, \chi)^4 L(s, \overline{\chi})^4 L(s, \chi^2) L(s, \overline{\chi}^2).$$

Moreover, in the special case when $\pi = \mathbb{1}$, $\rho = \psi \in \mathfrak{F}_1$ is quadratic and $\chi \in \mathfrak{F}_1$ is quadratic,

$$D(s) = \zeta(s)^4 L(s, \psi)^4 L(s, \chi)^4 L(s, \psi\chi)^4.$$

So our auxiliary L-function $D(s)$ generalizes the classical auxiliary L-functions used by de la Vallée Poussin (1899) and Siegel (1935). We only observed this in retrospect...

Surprise: Generalizing Tatuzawa's theorem

Jesse Thorner visited me the past week (he is flying back home right now), and it seems that we can prove the following generalization of Tatuzawa's theorem (over any fixed number field).

Theorem (Harcos–Thorner, announcement only)

For every $\varepsilon > 0$ and $n, m \in \mathbb{Z}_{\geqslant 1}$, there exists an effective constant $c_9 = c_9(n, m, \varepsilon) > 0$ with the following property.

Let $(\pi, \rho) \in \mathfrak{F}_n \times \mathfrak{F}_m$. Then there exists $\widetilde{\chi} \in \mathfrak{F}_1$ such that, for every $\chi \in \mathfrak{F}_1$, either

$$L(s, \pi \times (\rho \otimes \chi)) = L(s + it, \pi \times (\rho \otimes \widetilde{\chi}))$$

holds for some $t \in (-\varepsilon, \varepsilon)$, or

$$L(\sigma, \pi \times (\rho \otimes \chi)) \neq 0, \qquad \sigma \geqslant 1 - c_9(C(\pi)C(\rho)C(\chi))^{-\varepsilon}.$$