STIRLING'S APPROXIMATION

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Theorem 1. *For every positive integer n we have*

$$\frac{4^n}{\sqrt{\pi(n+1)}} < \binom{2n}{n} < \frac{4^n}{\sqrt{\pi n}}.$$

Proof. We present a proof due to Noam D. Elkies [1] and David E. Speyer [2]. The starting point is the identity

$$\begin{aligned}\int_{-\pi/2}^{\pi/2}(\cos x)^{2n}\,dx &= \int_{-\pi/2}^{\pi/2}\left(\frac{e^{ix}+e^{-ix}}{2}\right)^{2n}dx\\ &= \frac{1}{4^n}\sum_{k=0}^{2n}\binom{2n}{k}\int_{-\pi/2}^{\pi/2}e^{i(2n-2k)x}\,dx\\ &= \frac{\pi}{4^n}\binom{2n}{n}.\end{aligned}$$

By this identity, it suffices to show that

$$\sqrt{\frac{\pi}{n+1}} < \int_{-\pi/2}^{\pi/2}(\cos x)^{2n}\,dx < \sqrt{\frac{\pi}{n}}.$$

For the lower bound, we rewrite the integral in terms of $u := \tan x$, and estimate it as

$$\int_{-\pi/2}^{\pi/2}(\cos x)^{2n}\,dx = \int_{-\infty}^{\infty}\frac{1}{(1+u^2)^{n+1}}\,du > \int_{-\infty}^{\infty}e^{-(n+1)u^2}\,du = \sqrt{\frac{\pi}{n+1}}.$$

Here we used that $1+t<e^t$ for $t\neq 0$, which is a consequence of the strict convexity of the function $t\mapsto e^t$ on $\mathbb{R}$. For the upper bound, we estimate the integral as

$$\int_{-\pi/2}^{\pi/2}(\cos x)^{2n}\,dx < \int_{-\pi/2}^{\pi/2}e^{-nx^2}\,dx < \int_{-\infty}^{\infty}e^{-nx^2}\,dx = \sqrt{\frac{\pi}{n}}.$$

Here we used that $\cos x < e^{-x^2/2}$ for $0<|x|<\pi/2$, which is a consequence of the strict concavity of the function $x\mapsto \log\cos x + x^2/2$ on $(-\pi/2,\pi/2)$. $\square$

Theorem 2. *We have, as $n\to\infty$,*

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Proof. Let $n\to\infty$. First we prove the weaker statement that

$$n! \sim c\sqrt{n}\left(\frac{n}{e}\right)^n \tag{1}$$

holds for some constant $c>0$. Equivalently,

$$\sum_{k=1}^{n}\log k = \left(n+\frac{1}{2}\right)\log n - n + \log c + o(1). \tag{2}$$

To prove this, we write the left-hand side as a Riemann–Stieltjes integral (using $\log 1 = 0$)

$$\sum_{k=1}^{n}\log k = \int_1^n \log x\,d[x],$$

where $x \mapsto [x]$ is the integer part function. We decompose

$$[x] = \left(x - \frac{1}{2}\right) - \left(\{x\} - \frac{1}{2}\right),$$

where $x \mapsto \{x\}$ is the fractional part function, then we get

$$\sum_{k=1}^{n} \log k = \int_1^n \log x \, dx - \int_1^n \log x \, d\left(\{x\} - \frac{1}{2}\right).$$

On the right-hand side, we evaluate the first integral explicitly, and we apply integration by parts on the second integral. We obtain

$$\sum_{k=1}^{n} \log k = n(\log n - 1) + 1 + \frac{\log n}{2} + \int_1^n \frac{\{x\} - \frac{1}{2}}{x} \, dx.$$

We estimate the integral on the right-hand side by rewriting it as

$$\int_1^n \frac{\{x\} - \frac{1}{2}}{x} \, dx = \int_1^n \frac{dS(x)}{x} = \frac{S(n)}{n} + \int_1^n \frac{S(x)}{x^2} \, dx,$$

where

$$S(x) := \int_1^x \left(\{t\} - \frac{1}{2}\right) dt = \frac{\{x\}^2 - \{x\}}{2}.$$

It is clear that $S(x)$ is bounded (in fact $-1/8 \leqslant S(x) \leqslant 0$), therefore we infer

$$\int_1^n \frac{\{x\} - \frac{1}{2}}{x} \, dx = o(1) + \int_1^\infty \frac{S(x)}{x^2} \, dx - \int_n^\infty \frac{S(x)}{x^2} \, dx = \int_1^\infty \frac{S(x)}{x^2} \, dx + o(1),$$

where the last improper integral converges. Choosing $c > 0$ such that $-1 + \log c$ equals this improper integral, we conclude (2):

$$\begin{aligned}\sum_{k=1}^{n} \log k &= n(\log n - 1) + 1 + \frac{\log n}{2} + \int_1^n \frac{\{x\} - \frac{1}{2}}{x} \, dx \\ &= n(\log n - 1) + 1 + \frac{\log n}{2} - 1 + \log c + o(1) \\ &= \left(n + \frac{1}{2}\right) \log n - n + \log c + o(1).\end{aligned}$$

We have established (1), and it remains to show that in this asymptotic formula the constant c equals $\sqrt{2\pi}$. To see this, we observe the following consequence of (1):

$$\binom{2n}{n} = \frac{(2n)!}{n!^2} \sim \frac{c\sqrt{2n}\left(\frac{2n}{e}\right)^{2n}}{c^2 n \left(\frac{n}{e}\right)^{2n}} = 4^n \frac{\sqrt{2}}{c\sqrt{n}}.$$

However, by Theorem 1,

$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}.$$

Comparing the right-hand sides, the claim $c = \sqrt{2\pi}$ follows. □

References

[1] N. D. Elkies, Response to MathOverflow question No. 133732, https://mathoverflow.net/questions/133732

[2] D. E. Speyer, Response to MathOverflow question No. 133732, https://mathoverflow.net/questions/133732

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