SUMMARY OF UNDERGRADUATE THESIS

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Let m be a Markov number, i.e. a solution of the diophantine equation

$$m_1^2 + m_2^2 + m_3^2 = 3m_1 m_2 m_3.$$

Let $\mathcal{M}_m$ denote the set of x satisying $F(x,1) = 0$ for some quadratic form F corresponding to m. Write $\mathcal{M}$ for the union of the various $\mathcal{M}_m$. See [1] for details. Call two real numbers equivalent if they belong to the same orbit under the action of $GL_2(\mathbb{Z})$ by fractional linear transformations. Also, call two real numbers strongly equivalent if their sum or difference is an integer. Finally, introduce the functions

$$\nu(\xi) \stackrel{\mathrm{df}}{=} \liminf_{q\to\infty} q\|q\xi\|,$$

$$\chi(\xi) \stackrel{\mathrm{df}}{=} \inf_{q>0} q\|q\xi\|.$$

I prove an analogue for χ of a well-known result about the spectrum of ν. I learned only later that the result also appeared in Gurwood's unpublished Ph.D. thesis [2]. However, the proofs are different.

Theorem 1 (Perron [4], Heawood [3], Shibata [5]).

(1) *If $\nu(\xi) > \frac{1}{3}$ then ξ is equivalent to an element of $\mathcal{M}$.*
(2) *Conversely, if ξ is equivalent to an element of $\mathcal{M}_m$ then*

$$\nu(\xi) = \frac{1}{\sqrt{9-4m^{-2}}} > \frac{1}{3},$$

and the inequality $q\|q\xi\| < \nu(\xi)$ has infinitely many integer solutions q.
(3) *There are continuum many pairwise inequivalent ξ with $\nu(\xi) = \frac{1}{3}$.*

Theorem 2.

(1) *If $\chi(\xi) > \frac{1}{3}$ then ξ is strongly equivalient to an element of $\mathcal{M}$.*
(2) *Conversely, if ξ is strongly equivalent to an element of $\mathcal{M}_m$ then*

$$\chi(\xi) = \frac{2}{3+\sqrt{9-4m^{-2}}} > \frac{1}{3},$$

and the equation $q\|q\xi\| = \chi(\xi)$ has the only positive integer solution $q = m$.
(3) *There are continuum many pairwise inequivalent ξ with $\chi(\xi) = \frac{1}{3}$.*

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References

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3. Heawood, P., *The classification of rational approximation*, Proc. London Math. Soc. **20** (1922), 233–250.
4. Perron, O., *Über die Approximation irrationaler Zahlen durch rationale*, S. B. Heidelberg Akad. Wiss. (1921).
5. Shibata, K., *On the approximation of irrational numbers by rational numbers*, Japanese Journal of Math. **1** (1924), 9–13; Tohoku Math. Journal **23** (1924), 328–337.