On the sup-norm of Maass cusp forms of large level

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Comparing the various Sobolev norms of automorphic forms is useful in the theory of quantum chaos and subconvexity of L-functions, which in turn have deep arithmetic applications. We consider the following special case.

Problem. *Let f be a Hecke–Maass cuspidal newform of level N and Laplacian eigenvalue λ. Assume that $\|f\|_2 = 1$ with respect to $dxdy/y^2$. Bound $\|f\|_\infty$ in terms of N and λ.*

In the λ-aspect the first nontrivial (and so far unsurpassed) bound is due to Iwaniec and Sarnak [6]: $\|f\|_\infty \ll_{N,\epsilon} \lambda^{5/24+\epsilon}$ for any $\epsilon > 0$. In the N-aspect the trivial bound is $\|f\|_\infty \ll_{\lambda,\epsilon} N^{\epsilon}$, while the most optimistic bound would be $\|f\|_\infty \ll_{\lambda,\epsilon} N^{-1/2+\epsilon}$. Here and later, the dependence on λ is understood to be continuous. The breakthrough in the N-aspect was recently achieved by Blomer–Holowinsky [2] who proved $\|f\|_\infty \ll_{\lambda,\epsilon} N^{-25/914+\epsilon}$, at least for square-free N. The restriction on N seems difficult to remove as it is needed for a certain application of Atkin–Lehner theory. By a systematic use of geometric arguments Templier [7] derived $\|f\|_\infty \ll_{\lambda,\epsilon} N^{-1/22+\epsilon}$, and Helfgott–Ricotta [3] improved this to $\|f\|_\infty \ll_{\lambda,\epsilon} N^{-1/20+\epsilon}$. As we shall explain below, an efficient use of Atkin–Lehner theory leads to a short and clean proof of the following result [5]:

Theorem. *Let f be an L^2-normalized Hecke–Maass cuspidal newform of square-free level N, trivial nebentypus, and Laplacian eigenvalue λ. Then for any $\epsilon > 0$ we have a bound*

$$\|f\|_\infty \ll_{\lambda,\epsilon} N^{-1/6+\epsilon},$$

where the implied constant depends continuously on λ.

The theorem improves our earlier bound [4] with exponent $-1/12+\epsilon$. A hybrid version can also be established, improving significantly on [2, Theorem 2].

We turn to an informal discussion of our method. Very vaguely, the idea of proving a result as above has been like this:

(1) Pick any $z \in \mathfrak{H}$ where $|f(z)|$ needs to be estimated.
(2) Apply an Atkin–Lehner operator on z to ensure that $\operatorname{Im} z$ is not too small.
(3) Use the amplification method and some trace formula to reduce the problem to a counting problem depending on z.
(4) Do the counting based on the diophantine properties of z.

Our improvement results mainly from the following shortcut:

(2') Apply an Atkin–Lehner operator on z to maximize $\operatorname{Im} z$.
(4') Observe that z has good diophantine properties automatically, allowing a more efficient counting.

For a square-free level N the Atkin–Lehner operators can be represented by matrices of the form

$$W_M = \frac{1}{\sqrt{M}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}), \qquad M \mid N,$$

where $a, b, c, d \in \mathbb{Z}$ are integers satisfying

$$ad - bc = M, \qquad a \equiv 0\,(M), \qquad d \equiv 0\,(M), \qquad c \equiv 0\,(N).$$

A key feature is the multiplication rule

$$W_M W_{M'} = W_{M''} \qquad \text{with} \qquad M'' = \frac{MM'}{(M, M')^2},$$

which shows that the W_M's form a group $A_0(N)$ containing $\Gamma_0(N)$ as a normal subgroup. As a result, Atkin–Lehner operators induce an action on $\Gamma_0(N)\backslash\mathfrak{H}$ by the finite group $A_0(N)/\Gamma_0(N) \cong (\mathbb{Z}/2\mathbb{Z})^{\omega(N)}$, where $\omega(N)$ is the number of distinct prime factors of N.

By Atkin–Lehner theory [1], a Hecke–Maass cuspidal newform f of level N is an eigenvector for $A_0(N)$ with eigenvalues ± 1, therefore in examining the sup-norm of f we can restrict to the following fundamental domain for $A_0(N)$:

$$\mathcal{F}(N) := \{z \in \mathfrak{H} \mid \operatorname{Im} z \geqslant \operatorname{Im} \delta z \text{ for all } \delta \in A_0(N)\}.$$

Our starting point was the observation that the elements of $\mathcal{F}(N)$ have good diophantine properties (we assume that N is square-free):

Lemma. *Let $z = x + iy \in \mathcal{F}(N)$. Then the lattice $\langle 1, z\rangle$ has minimal distance at least $N^{-1/2}$ and covolume $y \gg N^{-1}$.*

The usefulness of this lemma becomes apparent when we relate $|f(z)|$ to a lattice counting problem depending on z. By combining the amplification method of Duke–Friedlander–Iwaniec with the pretrace formula of Selberg we obtain

$$\Lambda^2 \, |f(z)|^2 \ll_{\lambda,\epsilon} N^\epsilon \sum_{l \geqslant 1} \frac{y_l}{\sqrt{l}} M(z, l, N),$$

where $M(z, l, N)$ denotes the number of integral matrices $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$(*) \qquad \det(\gamma) = l, \qquad c \equiv 0\,(N), \qquad \left|-cz^2 + (a-d)z + b\right|^2 \leqslant l y^2 N^\epsilon,$$

Λ is a large parameter (the amplifier length), and

$$y_l := \begin{cases} \Lambda, & l = 1; \\ 1, & l = l_2 \text{ or } l_1 l_2 \text{ or } l_1 l_2^2 \text{ or } l_1^2 l_2^2 \text{ with } \Lambda < l_1, l_2 < 2\Lambda \text{ primes}; \\ 0, & \text{otherwise}. \end{cases}$$

Our second key observation is that for each range $l \asymp L$ and for each fixed c the inequality in $(*)$ can be used to bound the number of choices for the pair $(a-d, b)$.

Indeed, the pairs correspond to the lattice points in a disk of radius $\ll L^{1/2}yN^\epsilon$, hence the Lemma together with some geometry of numbers yields the bound

$$\#(a-d,b) \ll_\epsilon N^\epsilon \left(1 + L^{1/2}N^{1/2}y + Ly\right)$$

for any c. A simple manipulation of $(*)$ also shows $c \ll_\epsilon L^{1/2}N^\epsilon/y$. Finally for any triple $(c, a-d, b)$ we regard the identity

$$(a+d)^2 - 4l = (a-d)^2 + 4bc$$

as an equation for the pair $(a+d, l)$. By the sparsity of potential l's we can bound the number of quadruples $(c, a-d, b, a+d)$ efficiently, which of course is the same as bounding the sum of $M(z,l,N)$ over the l's considered.

Along these lines we obtain

$$\Lambda^2\left|f(z)\right|^2 \ll_{\lambda,\epsilon} N^\epsilon\left(\Lambda + \Lambda^{5/2}N^{-1/2} + \Lambda^4 N^{-1}\right),$$

at least when $y < N^{-2/3}$ and $\Lambda^4 < y^{-2}N^{-\epsilon}$. The latter is automatic for $y < N^{-2/3}$ under the choice

$$\Lambda := N^{1/3-\epsilon},$$

which incidentally also balances the terms in the previous display. Hence by amplification we really see that

$$f(x+iy) \ll_{\lambda,\epsilon} N^{-1/6+\epsilon}, \qquad y < N^{-2/3}.$$

For the remaining range $y \geqslant N^{-2/3}$ we use a simple bound based on the Fourier expansion at the cusp ∞ (see [7, § 3.2] or [2, (92) & (27)]):

$$f(x+iy) \ll_{\lambda,\epsilon} N^{-1/2+\epsilon}y^{-1/2} \leqslant N^{-1/6+\epsilon}, \qquad y \geqslant N^{-2/3}.$$

References

[1] A. O. L. Atkin, J. Lehner, *Hecke operators on* $\Gamma_0(m)$, Math. Ann. **185** (1970), 134–160.

[2] V. Blomer, R. Holowinsky, *Bounding sup-norms of cusp forms of large level*, Invent. Math. **179** (2010), 645–681.

[3] H. Helfgott, G. Ricotta, *A new bound for the sup norm of automorphic forms on non-compact modular curves in the level aspect*, unpublished

[4] G. Harcos, N. Templier, *On the sup-norm of Maass cusp forms of large level. II*, Int. Math. Res. Notices, to appear

[5] G. Harcos, N. Templier, *On the sup-norm of Maass cusp forms of large level. III*, under preparation

[6] H. Iwaniec, P. Sarnak, L^∞ *norms of eigenfunctions of arithmetic surfaces*, Ann. of Math. **141** (1995), 301–320.

[7] N. Templier, *On the sup-norm of Maass cusp forms of large level*, Selecta Math. (N.S.) **16** (2010), 501–531.