ON THE HYBRID MOMENTS OF THE GENERALIZED DIVISOR FUNCTION

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The purpose of these notes is to provide an elementary upper bound for the hybrid moments and the size of the generalized divisor function

$$\tau_k(n) := \sum_{n=n_1\dots n_k} 1.$$

Lemma 1. *Given a decomposition $mn = c_1\dots c_k$, there are decompositions $m = a_1\dots a_k$ and $n = b_1\dots b_k$ satisfying $c_i = a_i b_i$ ($1 \leqslant i \leqslant k$).*

Proof. We induct on k. For $k = 1$ the statement is clear upon writing $a_1 := m$ and $b_1 := n$. Now we assume that $k \geqslant 2$ and the statement is valid with $k-1$ in place of k. Given a decomposition $mn = c_1\dots c_k$, we write

$$a_k := (m, c_k), \qquad b_k := c_k/a_k, \qquad m' := m/a_k, \qquad n' := n/b_k.$$

These are positive integers satisfying $c_k = a_k b_k$ and $m'n' = c_1\dots c_{k-1}$. By the induction hypothesis, there are decompositions $m' = a_1\dots a_{k-1}$ and $n' = b_1\dots b_{k-1}$ with $c_i = a_i b_i$ ($1 \leqslant i \leqslant k-1$), and hence $m = a_1\dots a_k$ and $n = b_1\dots b_k$ with $c_i = a_i b_i$ ($1 \leqslant i \leqslant k$). □

Lemma 2. *For any integers $k, m, n \geqslant 1$, we have $\tau_k(mn) \leqslant \tau_k(m)\tau_k(n)$.*

Proof. This follows immediately from Lemma 1, upon noting that the assignment

$$(c_1,\dots,c_k) \mapsto ((a_1,\dots,a_k),(b_1,\dots,b_k))$$

is injective by $c_i = a_i b_i$ ($1 \leqslant i \leqslant k$). □

Remark 1. Alternatively, Lemma 2 follows from the fact that τ_k is multiplicative with values at the the prime powers given by

$$\tau_k(p^\nu) = \binom{\nu+k-1}{k-1} = \prod_{j=1}^{k-1}\left(1+\frac{\nu}{j}\right).$$

Theorem 1. *For any integers $k_1,\dots,k_\ell \geqslant 1$ and any real number $x \geqslant 1$, we have*

$$\sum_{n\leqslant x}\frac{\tau_{k_1}(n)\dots\tau_{k_\ell}(n)}{n} \leqslant \left(\sum_{n\leqslant x}\frac{1}{n}\right)^{k_1\dots k_\ell} \tag{1}$$

and

$$\sum_{n\leqslant x}\tau_{k_1}(n)\dots\tau_{k_\ell}(n) \leqslant x\left(\sum_{n\leqslant x}\frac{1}{n}\right)^{k_1\dots k_\ell-1}. \tag{2}$$

Proof. First we prove (1) by induction on the number of factors ℓ. For $\ell = 0$ the statement is trivial, upon regarding the empty product to be 1. Now we assume that $\ell \geqslant 1$ and (1) is valid with $\ell-1$ in place of ℓ. By Lemma 2, we have in general

$$\tau_k(n_1\dots n_j) \leqslant \tau_k(n_1)\dots\tau_k(n_j). \tag{3}$$

We shall apply this for $j := k_\ell$ and each $k \in \{k_1, \dots, k_{\ell-1}\}$. Then, using also the induction hypothesis,

$$\begin{aligned}\sum_{n\leqslant x}\frac{\tau_{k_1}(n)\dots\tau_{k_\ell}(n)}{n} &= \sum_{n_1\dots n_{k_\ell}\leqslant x}\frac{\tau_{k_1}(n_1\dots n_{k_\ell})\dots\tau_{k_{\ell-1}}(n_1\dots n_{k_\ell})}{n_1\dots n_{k_\ell}}\\ &\leqslant \sum_{n_1\dots n_{k_\ell}\leqslant x}\prod_{i=1}^{k_\ell}\frac{\tau_{k_1}(n_i)\dots\tau_{k_{\ell-1}}(n_i)}{n_i}\\ &\leqslant \left(\sum_{n\leqslant x}\frac{\tau_{k_1}(n)\dots\tau_{k_{\ell-1}}(n)}{n}\right)^{k_\ell}\\ &\leqslant \left(\sum_{n\leqslant x}\frac{1}{n}\right)^{k_1\dots k_\ell}.\end{aligned}$$

The proof of (1) is complete.

We prove (2) similarly. For $\ell = 0$ the statement is trivial, hence we assume that $\ell \geqslant 1$ and (2) is valid with $\ell - 1$ in place of ℓ. Using (3) as before, we see that

$$\begin{aligned}\sum_{n\leqslant x}\tau_{k_1}(n)\dots\tau_{k_\ell}(n) &= \sum_{n_1\dots n_{k_\ell}\leqslant x}\tau_{k_1}(n_1\dots n_{k_\ell})\dots\tau_{k_{\ell-1}}(n_1\dots n_{k_\ell})\\ &\leqslant \sum_{n_1\dots n_{k_\ell}\leqslant x}\prod_{i=1}^{k_\ell}\tau_{k_1}(n_i)\dots\tau_{k_{\ell-1}}(n_i).\end{aligned}$$

The right hand side equals

$$\sum_{n_1\dots n_{k_\ell-1}\leqslant x}\prod_{i=1}^{k_\ell-1}\tau_{k_1}(n_i)\dots\tau_{k_{\ell-1}}(n_i)\sum_{n\leqslant x/(n_1\dots n_{k_\ell-1})}\tau_{k_1}(n)\dots\tau_{k_{\ell-1}}(n),$$

and the inner sum can be estimated by the induction hypothesis as

$$\sum_{n\leqslant x/(n_1\dots n_{k_\ell-1})}\tau_{k_1}(n)\dots\tau_{k_{\ell-1}}(n) \leqslant \frac{x}{n_1\dots n_{k_\ell-1}}\left(\sum_{n\leqslant x}\frac{1}{n}\right)^{k_1\dots k_{\ell-1}-1}.$$

It follows that

$$\begin{aligned}\sum_{n\leqslant x}\tau_{k_1}(n)\dots\tau_{k_\ell}(n) &\leqslant x\left(\sum_{n\leqslant x}\frac{1}{n}\right)^{k_1\dots k_{\ell-1}-1}\left(\sum_{n_1\dots n_{k_\ell-1}\leqslant x}\prod_{i=1}^{k_\ell-1}\frac{\tau_{k_1}(n_i)\dots\tau_{k_{\ell-1}}(n_i)}{n_i}\right)\\ &\leqslant x\left(\sum_{n\leqslant x}\frac{1}{n}\right)^{k_1\dots k_{\ell-1}-1}\left(\sum_{n\leqslant x}\frac{\tau_{k_1}(n)\dots\tau_{k_{\ell-1}}(n)}{n}\right)^{k_\ell-1}.\end{aligned}$$

The rightmost sum can be estimated via (1), and we infer that

$$\sum_{n\leqslant x}\tau_{k_1}(n)\dots\tau_{k_\ell}(n) \leqslant x\left(\sum_{n\leqslant x}\frac{1}{n}\right)^{k_1\dots k_{\ell-1}-1+k_1\dots k_{\ell-1}(k_\ell-1)} = x\left(\sum_{n\leqslant x}\frac{1}{n}\right)^{k_1\dots k_\ell-1}.$$

The proof of (2) is complete. □

Theorem 2. *For any integers $k, \ell \geqslant 1$ and any real number $x \geqslant 1$, we have*

$$\sum_{n\leqslant x}\frac{\tau_k(n)^\ell}{n} \leqslant \left(\sum_{n\leqslant x}\frac{1}{n}\right)^{k^\ell} \tag{4}$$

and

$$\sum_{n\leqslant x}\tau_k(n)^\ell \leqslant x\left(\sum_{n\leqslant x}\frac{1}{n}\right)^{k^\ell-1}. \tag{5}$$

Proof. This theorem is the special case $k_1=\cdots=k_\ell$ of Theorem 1. □

Remark 2. Theorem 2 can also be proved directly, by induction on ℓ (starting from the base case $\ell=0$). This proof is much the same as the proof of Theorem 1, but it is notationally simpler.

Theorem 3. *We have a uniform upper bound*

$$\tau_k(n)\leqslant n^{\frac{\log k}{\log\log n}+O_k\left(\frac{\log\log\log n}{(\log\log n)^2}\right)}.$$

Proof. By (5) we have, for any integer $\ell\geqslant 1$,

$$\log\tau_k(n)\leqslant\frac{\log n}{\ell}+\frac{k^\ell-1}{\ell}\log(1+\log n)=\frac{\log n}{\ell}+O\left(\frac{k^\ell}{\ell}\log\log n\right).$$

We can choose ℓ so that

$$\frac{k^\ell}{\ell}\asymp_k\frac{\log n}{(\log\log n)^3}.$$

Then a quick calculation gives

$$\ell\log k=\log\log n-2\log\log\log n+O_k(1),$$

whence

$$\frac{\log\tau_k(n)}{\log n}\leqslant\frac{1}{\ell}+O_k\left(\frac{1}{(\log\log n)^2}\right)=\frac{\log k}{\log\log n}+O_k\left(\frac{\log\log\log n}{(\log\log n)^2}\right).$$

□

Remark 3. Theorem 3 in the special case of $k=2$ was proved by Wigert [3] in 1906. He used the Prime Number Theorem in his argument, which is also recorded in Landau's *Handbuch* [1, §60]. A few years later, Ramanujan [2] observed the elementary character of the inequality, and in fact our proof above uses only Euclid's Algorithm and its immediate consequences. However, as Ramanujan [2] discusses, the error term has a strong connection to the distribution of prime numbers, hence better estimates can be proved or remain conjectured.

REFERENCES

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