A new zero-free region for Rankin–Selberg L-functions

Gergely Harcos
(joint work with Jesse Thorner)

Establishing zero-free regions for automorphic L-functions is a central problem of number theory. We report about the recent preprint of the same title [3], which extends Siegel's celebrated lower bound for Dirichlet L-functions [15] to all GL_1-twists of $\mathrm{GL}_n \times \mathrm{GL}_{n'}$ Rankin–Selberg L-functions. The result is meant over an arbitrary number field, but for simplicity we restrict the present summary to the rational field.

Let $\mathfrak{F}_n$ be the set of unitary cuspidal representations of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$. Let $\mathfrak{F}_n^* \subset \mathfrak{F}_n$ be the set of unitary cuspidal representations of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ whose central character is trivial on $\mathbb{R}_{>0}$. In particular, $\mathfrak{F}_1$ is the group of unitary Hecke characters of $\mathbb{A}_\mathbb{Q}^\times \cong \mathbb{R}_{>0} \times \mathbb{A}_\mathbb{Q}^1$, which acts on $\mathfrak{F}_n$ via

$$(\pi \otimes \chi)(g) = \pi(g)\chi(\det g).$$

It follows that, for each $\pi \in \mathfrak{F}_n$, there is a unique pair $(t_\pi, \pi^*) \in \mathbb{R} \times \mathfrak{F}_n^*$ satisfying

$$\pi = \pi^* \otimes |\cdot|^{it_\pi}, \qquad L(s,\pi) = L(s+it_\pi, \pi^*).$$

Moreover, for all $(\pi,\pi') \in \mathfrak{F}_n \times \mathfrak{F}_{n'}$,

$$L(s, \pi \times \pi') = L(s + it_\pi + it_{\pi'}, \pi^* \times \pi'^*).$$

The point of this normalization is that $L(s,\pi^*)$ and $L(s,\pi^* \times \pi'^*)$ can only have a pole at $s=1$, hence $L(s,\pi)$ and $L(s,\pi\times\pi')$ can only have a pole at $s = 1 - it_\pi$ and $s = 1 - it_\pi - it_{\pi'}$, respectively. Finally, we shall denote the analytic conductor of $L(s,\pi)$ by $C(\pi)$.

The classical results of de la Vallée Poussin [12] and Siegel [15] have been extended to the standard L-functions $L(s,\pi)$ considered above. Important milestones include Jacquet–Shalika [8], Moreno [11], and Hoffstein–Ramakrishnan [4]. See Brumley [5, Theorem A.1] and Jiang–Lü–Thorner–Wang [9, Section 4] for important recent results.

According to the Langlands functoriality conjecture, the Rankin–Selberg L-function $L(s,\pi\times\pi')$ equals a product of standard L-functions $L(s,\Pi)$. However, we are far from knowing this, and correspondingly, we understand the analytic properties of Rankin–Selberg L-functions much less than those of standard L-functions. Concerning non-vanishing, Shahidi [14, Theorem 5.2] proved that $L(s,\pi\times\pi') \neq 0$ for $\Re(s) \geq 1$. The first uniform improvement over this basic result is due to Brumley (see [1] and [10, Theorem A.1]); we display slightly weaker exponents for notational simplicity:

Theorem 1. *There exists a constant $c_1 = c_1(n,n') > 0$ with the following property. If $(\pi,\pi') \in \mathfrak{F}_n^* \times \mathfrak{F}_{n'}^*$, then $L(s,\pi\times\pi')$ has no zero in the region*

$$\Re(s) \geq 1 - c_1(C(\pi)C(\pi'))^{-n-n'}(|\mathrm{Im}(s)|+1)^{-nn'}.$$

The classical results of de la Vallée Poussin [12] and Siegel [15] have also been extended to special Rankin–Selberg L-functions. Important milestones include Moreno [11], Sarnak [13], Goldfeld–Li [2], and Humphries [5]. As a combination of Brumley [5, Theorem A.1] and Humphries–Thorner [7, Theorem 2.1], we have

Theorem 2. *There exists $c_2 = c_2(n, n') > 0$ with the following property. If $(\pi, \pi') \in \mathfrak{F}_n^* \times \mathfrak{F}_{n'}^*$ satisfies $\pi = \tilde{\pi}$ or $\pi' = \tilde{\pi}'$ or $\pi' = \tilde{\pi}$, then $L(s, \pi \times \pi')$ has at most one zero β (necessarily real and simple) in*

$$\Re(s) \geq 1 - c_2/\log(C(\pi)C(\pi')(|\mathrm{Im}(s)| + 3)).$$

If the exceptional zero β exists, then $(\pi, \pi') = (\tilde{\pi}, \tilde{\pi}')$ or $\pi' = \tilde{\pi}$.

The next result we highlight is due to Humphries–Thorner [6, Theorem 2.4].

Theorem 3. *For every $\pi \in \mathfrak{F}_n^*$ and $\varepsilon > 0$, there exists $c_3 = c_3(\pi, \varepsilon) > 0$ with the following property. If $\chi \in \mathfrak{F}_1^*$ is quadratic, then*

$$L(\sigma, \pi \otimes (\tilde{\pi} \otimes \chi)) \neq 0, \qquad \sigma \geq 1 - c_3 C(\chi)^{-\varepsilon}.$$

In the recent preprint [3], we extended Siegel's celebrated lower bound for Dirichlet L-functions [15] to all GL_1-twists of $L(s, \pi \times \pi')$. The proof relies crucially on the group structure of $\mathfrak{F}_1$, not just $\mathfrak{F}_1^*$. As a result, we substantially improved on Brumley's lower bound in the GL_1-twist aspect, but the dependence on (π, π') is no longer effective.

Theorem 4. *Let $(\pi, \pi') \in \mathfrak{F}_n \times \mathfrak{F}_{n'}$. For all $\varepsilon > 0$, there exists an ineffective constant $c_4 = c_4(\pi, \pi', \varepsilon) > 0$ such that if $\chi \in \mathfrak{F}_1$, then*

$$|L(\sigma, \pi \times (\pi' \otimes \chi))| \geq c_4 C(\chi)^{-\varepsilon}, \qquad \sigma \geq 1 - c_4 C(\chi)^{-\varepsilon}.$$

In particular, there exists $c_5 = c_5(\pi, \pi', \varepsilon) > 0$ such that

$$|L(\sigma + it, \pi \times \pi')| \geq c_5(|t| + 1)^{-\varepsilon}, \qquad \sigma \geq 1 - c_5(|t| + 1)^{-\varepsilon}.$$

The new zero-free region allows us to prove an analogue of the Siegel–Walfisz theorem for Rankin–Selberg L-functions.

Theorem 5. *For $(\pi, \pi') \in \mathfrak{F}_n \times \mathfrak{F}_{n'}$, let $\Lambda_{\pi \times \pi'}(m)$ denote the m-th Dirichlet coefficient of $-L'(s, \pi \times \pi')/L(s, \pi \times \pi')$, and let*

$$\mathcal{M}_{\pi \times \pi'}(x) = \begin{cases} x^{1-iu}/(1 - iu), & \pi' = \tilde{\pi} \otimes |\cdot|^{iu}; \\ 0, & \text{otherwise.} \end{cases}$$

If $q \leq (\log x)^A$ is a positive integer coprime to the conductors of π and π', and $\gcd(a, q) = 1$, then

$$\sum_{\substack{m \leq x \\ m \equiv a \pmod{q}}} \Lambda_{\pi \times \pi'}(m) = \frac{\mathcal{M}_{\pi \times \pi'}(x)}{\varphi(q)} + O_{\pi, \pi', A}\left(\frac{x}{(\log x)^A}\right).$$

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