A PROBLEM ON ZERO SUBSUMS IN ABELIAN GROUPS

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Abstract. Let $a_1, a_2, \dots$ be elements of an abelian group such that a_m has order larger than m^m. Then the multiset of the a_m's can be partitioned into two parts which are free of zero subsums. The result was motivated by a question of Paul Erdős.

1. Introduction

The starting point of the paper was the following conjecture of Erdős, related to us by Prof. R. Freud.

Conjecture. *If the prime p is large enough in terms of k then any multiset of k nonzero residue classes modulo p can be partitioned into two parts such that in none of the parts does a zero subsum occur.*

The authors know about two different proofs of the conjecture, one by L. Pósa and an other by A. Pál (cf. Remark 1). However, neither proof has been published, and neither provides an effective admissible bound for p. Also, the conjecture admits a straightforward generalization to abelian groups in which one requires that the order of the elements involved be sufficiently large in terms of k. A justification for this conjecture might be that any torsion free abelian group can be ordered, whence its nonzero elements split into positive and negative parts. We obtain a common extension of all these generalized conjectures as follows.

Theorem. *Let a_m $(m = 1, 2, \dots)$ be elements of an abelian group A with a_m of order larger than m^m. Then the multiset of the a_m's can be partitioned into two zero subsum free parts.*

Clearly the result implies a similar statement for any finite sequence of the a_m's (by taking $A \oplus \mathbb{Z}$ in place of A), therefore we see now that $p > k^k$ is admissible in Erdős' question. In fact, a result of Bilu, Lev and Ruzsa [1] even shows that if $p > (2s)^k$ then any k-element set of mod p residues has a Freiman isomorphism into $\mathbb{Z}$, that is, a map which induces a bijection between s-term sums from the residues in question and their images. We mention here for sake of completeness that the exponential bound on p is essential, because any $2 < p \le 2^{(k-1)/2}$ would fail in the original conjecture (see Remark 2).

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2. Proof and remarks

In the following we say that an element a in an abelian group has order larger than m if $la \neq 0$ for any integer $1 \leq l \leq m$.

Proof of the Theorem. A well-known lemma of König on infinite trees shows that it suffices to find the required partition for each multiset $\{a_1, a_2, \dots, a_k\}$ ($k = 1, 2, \dots$). This we do by induction on k. Suppose that we have found a good partition for $k-1$ in place of k. This determines a partition $I_1 \cup I_2$ of the set $I = \{1, 2, \dots, k-1\}$ of suffices, and we want to find a similar decomposition of the set $L = \{1, 2, \dots k\}$.

As a_k has order larger than k^k, there exists a character $\varphi : A \to \mathbb{R}/\mathbb{Z}$ such that $\varphi(a_k)$ also has order larger than k^k. For sake of simplicity we represent the circle group $\mathbb{R}/\mathbb{Z}$ by the real interval $[-1/2, 1/2)$. We shall apply Dirichlet's theorem on simultaneous diophantine approximation to the k numbers $\varphi(a_m)$ ($m \in L$). This theorem yields, for any positive integer Q, an integer $1 \leq q \leq Q^k$ such that

$$\bigl|q\varphi(a_m)\bigr| < \frac{1}{Q} \quad (m \in L).$$

We take Q to be k; then all the $q\varphi(a_m)$ lie in $(-1/k, 1/k)$ and $q\varphi(a_k)$ is not zero by assumption on φ. In particular, the set

$$J = \bigl\{m \in L : q\varphi(a_m) = 0\bigr\}$$

is contained in I, and the partition $I_1 \cup I_2$ of I determines a partition $J_1 \cup J_2$ of J by $J_1 = J \cap I_1$ and $J_2 = J \cap I_2$. The complement of J in L is clearly a disjoint union $K_1 \cup K_2$ where

$$K_1 = \bigl\{m \in L : q\varphi(a_m) > 0\bigr\},$$

$$K_2 = \bigl\{m \in L : q\varphi(a_m) < 0\bigr\}.$$

Therefore we get a partition $L_1 \cup L_2$ of L by putting

$$L_1 = J_1 \cup K_1, \quad L_2 = J_2 \cup K_2.$$

We claim that the corresponding partition of the multiset $\{a_1, a_2, \dots, a_k\}$ consists of zero subsum free parts. Let M be any subset of L_1 or L_2. We consider the case $M \subseteq L_1$, the other case being analogous. Let

$$\sum_{m \in M} a_m = 0. \tag{1}$$

Then also

$$\sum_{m \in M} q\varphi(a_m) = 0,$$

whence

$$\sum_{m \in M \setminus J} q\varphi(a_m) = 0$$

by definition of J. As

$$M \setminus J \subseteq L_1 \setminus J = K_1,$$

all the terms involved in the last sum are strictly between 0 and $1/k$ by definition of K_1 and q. Since the number of terms is at most k, $M \setminus J$ must be empty. In other words, M is contained in J which gives

$$M \subseteq J_1 \subseteq I_1.$$

Now (1) combined with the property of I_1 shows that in fact $M = \emptyset$. This proves our claim, hence the theorem. $\square$

Remark 1. We can read off from the theorem or the proof itself that any k elements of orders larger than k^k in an abelian group can be divided into two zero subsum free parts. The existence of a constant depending on k like k^k in this statement can also be derived from Gödel's Compactness Theorem as follows. The Compactness Theorem reduces the existence of the constant depending on k to the statement that in an abelian group A any finite multiset of non-torsion elements can be partitioned into two zero subsum free parts. After passing from A to the quotient A/T where T is the subgroup of torsion elements in A it becomes sufficient to show that in a torsion free abelian group A any finite multiset of nonzero elements can be partitioned into two zero subsum free parts. That claim, however, is a consequence of the fact quoted above that A can be ordered. This is essentially the idea behind A. Pál's solution of the original Erdős problem.

Remark 2. Whenever an odd prime p is at most $2^{(k-1)/2}$, it fails to have the property in the Erdős Conjecture. To see this we can clearly assume that $(k-1)/2$ is an integer, say t. As $2 < p \leq 2^t$, we have $t \geq 2$. Consider the multiset of residues mod p consisting of two 1's, two -1's, one 2^s for each $s = 1, 2, \ldots t-1$ and one -2^s for each $s = 1, 2, \ldots t-2$. Altogether we have $4 + (t-1) + (t-2) = 2t + 1 = k$ nonzero residues modulo p. It is easy to see that in any partition of this multiset into zero subsum free parts the two 1's and the residues 2^s $(s = 1, 2, \ldots, t-1)$ are necessarily in one part. The subsums which can be formed of these are exactly the residues from 1 to 2^t. But then a zero subsum also occurs, because $p \leq 2^t$. Therefore the partition in question does not exist.

References

[1] Yu. Bilu, V. Lev and I. Z. Ruzsa, *Rectification principles in additive number theory*, J. Discrete Geo., to appear.

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