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ON RAMSEY COVERING-NUMBERS

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0. INTRODUCTION

In this paper I should like to sketch some aspects of a Ramsey-type problem (part 2.) which arose from a geometrical problem of T. Gallai [1]. Let me present the rough skeleton of the theorems discussed later.

If we colour the edges of a complete graph G with n colours in such a way that we need a sufficiently large number of one-coloured complete subgraphs of G in order to cover G's vertices then for at least one i, $(1 \leqslant i \leqslant n)$ G will contain a prescribed subgraph coloured with the i-th colour.

1. NOTIONS AND NOTATIONS

Graph	$G, H, \ldots$	finite, undirected, no loops and multiple edges
Vertex and edge set	$V(G), E(G)$	
Subgraph	$G \subset H$	*always* induced (spanned) subgraph

The set of n-coloured complete graphs	$\mathcal{K}(n)$	complete graphs the edges of which are coloured with n colours. It is *allowed* to colour an edge with more than one colour.
Covering number of an n-coloured complete graph G	$\alpha(G)$	it is the smallest k so that $V(G) = \bigcup_{i=1}^{k} V(G_i)$ and G_i is a one-coloured complete graph.
	$\mathcal{G}$	the set of all graphs
	$\mathcal{K}$	the set of complete graphs
	$\mathcal{H}^n$	denotes the Cartesian product of n copies of $\mathcal{H}$ where $\mathcal{H}$ is a set of graphs
	$\overline{\mathcal{H}}$	$= \{\bar{H}: H \in \mathcal{H}\}$ where $\mathcal{H}$ is a set of graphs

2. THE BASIC PROBLEM AND ITS RELATION TO THE ORIGINAL RAMSEY PROBLEM

2.1. According to a well-known theorem of Ramsey [2] for any $K = (K_1, K_2, \ldots, K_n) \in \mathcal{K}^n$ there exists a natural number $R = R(k)$ with the property:

$$\text{If} \quad G \in \mathcal{K}(n) \quad \text{and} \quad |V(G)| \geqslant R$$

then for at least one i, $(1 \leqslant i \leqslant n)$ G contains a subgraph in the i-th colour isomorphic to K_i. The smallest R with the above property is called the Ramsey-number belonging to $K_1, K_2, \ldots, K_n$ and it is denoted by $R_0(K)$.

Our basic problem is the following:

2.2. To describe the set $\mathcal{H} \subset \mathcal{G}^n$ for which the following statement holds

For every $H = (H_1, H_2, \ldots, H_n) \in \mathscr{H}$ *there exists a natural number* R_1 *with the property:*

If $G \in \mathscr{K}(n)$ *and* $\alpha(G) \geqslant R_1$ *then for at least one* i G *contains a subgraph in the* i*-th colour isomorphic to* H_i.

The smallest R_1 with the above property should be called the *Ramsey covering-number* belonging to H. It is denoted by $C_0 = C_0(H)$.

In case of $H \in \mathscr{K}^n$ the following trivial inequality holds between the Ramsey-numbers and the Ramsey covering-numbers:

$$C_0(H) \leqslant R_0(H) \leqslant \Big(\max_{1 \leqslant i \leqslant n} |V(H_i)| - 1\Big)(C_0(H) - 1) + 1 .$$

3. SOME RESULTS AND AN OPEN QUESTION

Let $H \subset \mathscr{G}^n$ be the set defined in 2.2. The following result was proved in [3].

Theorem 1. *Let* Q *be the set of graphs the complements of which contain no adjacent edges, then*

$$Q^n \subset \mathscr{H} .$$

Let us continue with the central open question:

Question 1. Let $\mathscr{R}$ be the set of graphs the complements of which contain no circles and let $\mathscr{A}(n)$ be the set of n-tuples formed by taking $n - 1$ components from $\mathscr{K}$ and one from $\mathscr{R}$. Is it true that

$$\mathscr{A}(n) \subset \mathscr{H} ?$$

(Special cases will be considered in part 4 and 5.)

There are some degenerate elements of $\mathscr{H}$.

(1) The n-tuples of graphs at least one component of which is the one-point graph or the two-point graph without edge.

(2) The n-tuples where at most one component differs from the two-point complete graph.

The n-tuples listed in (1) and (2) are called the degenerate elements of $\mathscr{H}$ and we denote them by $\mathscr{D}$.

Theorem 2. *If* $H = (H_1, \ldots, H_n) \notin \mathscr{A}(n) \cup Q^n \cup \mathscr{D}$ *then* $H \notin \mathscr{H}$.

Theorem 2 shows that the affirmative answer to Question 1 would settle the problem in 2.2.

4. STRONG COLOURINGS

Now we investigate the case when we colour the edges of the complete graphs with the restriction that every edge has exactly one colour. We call such a colouring "strong". The set of strongly n-coloured complete graphs will be denoted by $\mathscr{K}_s(n)$. One more notation: if $H = (H_1, \ldots, H_n)$ then $\bar{H}$ denotes $(\bar{H}_1, \ldots, \bar{H}_n)$. Let $\mathscr{H}_s$ be defined on the analogy of 2.2. if we write $\mathscr{K}_s(n)$ instead of $\mathscr{K}(n)$. It is obvious that $\mathscr{H} \subset \mathscr{H}_s$ and the following theorems show that the inclusion is proper.

Theorem 3. $\bar{Q}^n \subset \mathscr{H}_s$ *(Q defined in Theorem 1).*

Question 2. $\mathscr{A}(n) \cup \bar{\mathscr{A}}(n) \subset \mathscr{H}_s$? ($\mathscr{A}(n)$ defined in Question 1).

Theorem 4. *Let $\mathscr{B}_k$ be the set of complete k-partite graphs and T be the three-point graph with two edges. Let $\mathscr{T}(n)$ be the set of n-tuples with one component from $\mathscr{B}_k$ and the others are subgraphs of T. Then $\mathscr{T}(n) \subset \mathscr{H}_s$ and $\bar{\mathscr{T}}(n) \subset \mathscr{H}_s$.*

Theorem 5. *Let $\mathscr{L}$ be the set of graphs in the form $A \cup B$ where $A \cap B = \phi$, A is a complete graph and B is an at most one-point graph. Then $\mathscr{L}^n \subset \mathscr{H}_s$ and $\bar{\mathscr{L}}^n \subset \mathscr{H}_s$.*

Proposition 1. *If $\mathscr{D}^*$ denotes the n-tuples of graphs where at least two components are empty graphs, then $\mathscr{D}^* \subset \mathscr{H}_s$.*

Theorem 6. *If* $H \notin \mathscr{H} \cup \bar{Q}^n \cup \bar{\mathscr{A}}(n) \cup \mathscr{T}(n) \cup \bar{\mathscr{T}}(n) \cup \mathscr{L}^n \cup \bar{\mathscr{L}}^n \cup \cup \mathscr{D}^*$ *then* $\mathscr{H} \notin \mathscr{H}_s$.

5. PROPERTIES OF GRAPHS WITH LARGE CHROMATIC NUMBER AND WITHOUT COMPLETE k-GON

Let G be a graph. We may consider G as a strongly two-coloured complete graph by taking the edges of G $(\bar{G})$ as coloured with the first (second) colour. In this formulation the special case $n = 2$ of Question 2 is equivalent with

Question 3. Let F be a forest and k a natural number. Is there a natural number $l = l(F, k)$ with the property: if G is a graph without a complete k-gon and $\chi(G) \geqslant l$ then G containts F as a subgraph.

J. Gerlits proved first (oral communication) that the answer is affirmative to Question 3 if $k = 3$ and F is a path.

Let $\chi_0 = \chi_0(F, k)$ be the smallest number with the above property. Now we can state

Theorem 7. $\frac{|F| + 1}{2} \leqslant \chi_0(F, 3) \leqslant |F| - 1$ *(F is a path and $|F| \geqslant 4$).*

L. Lovász showed that $\chi_0(F, k)$ exists if F is a path for arbitrary k. The existence of $\chi_0(F, k)$ is proved otherwise only for $|F| \leqslant 5$ and $k = 3$ and for the (trivial) case when F is a star.

The upper bound in Theorem 7 follows from the following theorem.

Theorem 8. *Let G be a connected n-chromatic graph which contains no triangle and P and arbitrary point in G. There is a path of $n + 1$ points in G without diagonals.* $(n \geqslant 3)$.

6. HELLY STRUCTURES

A pair $(X, \mathscr{A})$ is called Helly structure if X is a set, $\mathscr{A}$ is a family of subsets of X and there exists a natural number t with the property:

If $\mathscr{B}$ is a finite subfamily of $\mathscr{A}$ any two members of $\mathscr{B}$ have non-empty intersection then there exists a set $P \subset X$ so that $|P| \leqslant t$ and $B \cap P \neq \phi$ if $B \in \mathscr{B}$.

Let $(X_1, \mathscr{A}_1), \ldots, (X_n, \mathscr{A}_n)$ be Helly structures and $X_i \cap X_j = \phi$ for $i \neq j$. We define the sum $\sum_{i=1}^{n} (X_i, \mathscr{A}_i) = (X, \mathscr{A})$ in the following way: $X = \bigcup_{i=1}^{n} X_i$, $\mathscr{A} = \{(A_1, A_2, \ldots, A_n): A_i \in \mathscr{A}_i\}$.

The following theorem connects the Helly structures and the set defined in 2.2.

Theorem 9. *Let $(X_1, \mathscr{A}_1) \ldots (X_n, \mathscr{A}_n)$ be Helly structures and suppose that the graph H_i can not be the intersection-graph of sets of $\mathscr{A}_i$ (for $1 \leqslant i \leqslant n$). In this case $(H_1, \ldots, H_n) \in \mathscr{H}$ implies that $\sum_{i=1}^{n} (X_i, \mathscr{A}_i)$ is also a Helly structure.*

Examples and applications of this theorem can be found in [3].

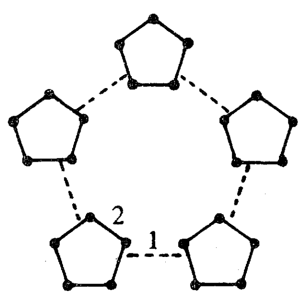
7. PROOFS

In this section we present the proofs of the theorems discussed above.

Theorem 1 was proved in [3].

For the proof of Theorem 2 we have to define some special m-coloured (or 2-coloured) complete graphs. The graph $U^k \in \mathscr{K}(m)$ looks like this:

U^3:

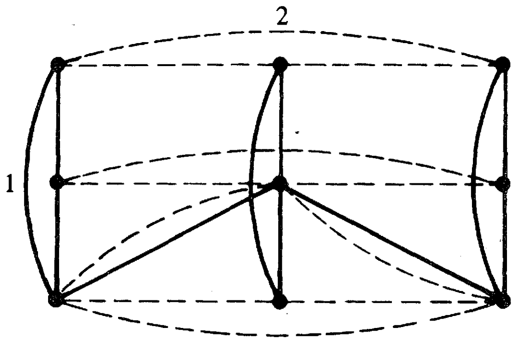


Let S^k be a graph containing no triangles and the chromatic number of which is k. Let $|V(S_n)| = n_k$ and A be a copy of S_k. Replace the vertices of A by $B_1, B_2, \ldots, B_{n_k}$ where B is a copy of S_k. All edges between B_i and B_j are coloured with colour 1 if the corresponding vertices of A are connected by an edge. The edges of B_i are coloured with colour 2 and all the remaining edges are coloured by $1, 2, \ldots$ and m.

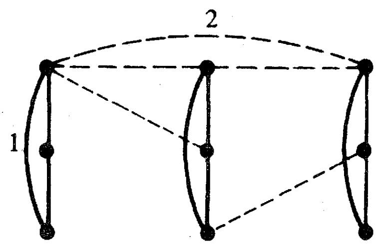
$W^k \in \mathscr{K}(2)$ is defined as follows: $V(W^k) = \{w_{ij}\}_{i,j=1}^{k}$. The edge connecting the vertices w_{ij} and w_{rs} is coloured with colour 1 if $i \neq r$ and coloured with colour 2 if $j \neq s$.

$W_1^k \in \mathscr{K}(2)$ has the points $\{w_{ij}\}_{i,j=1}^{k}$. The edge between w_{ij} and w_{rs} is coloured with 1 if $j = s$, otherwise it is coloured with colour 2.

W^3:



W_1^3:

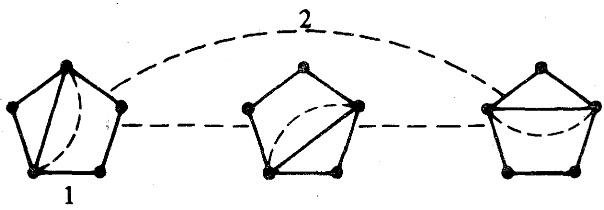


$X^k \in \mathscr{K}(2)$ will be defined as follows: $V(X^k) = \bigcup_{i=1}^{k} B_i$ where B_i is a copy of S_k. The edges of B_i are coloured with the colour 1, the edges between different B_i's have colour 2, the remaining edges are two-coloured.

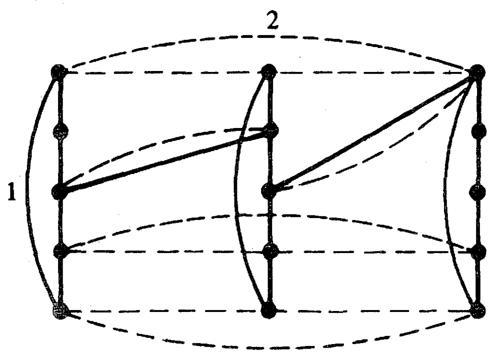
We define $Y^k \in \mathscr{K}(2)$: $V(Y^k) = \bigcup_{i=1}^{k} B_i$ where the B_i's are again copies of S_k. Let $V(B_i) = \{b_1^i, \dots, b_{n_k}^i\}$ and the edges of B_i have colour 1, the edge between b_t^i and b_t^j is coloured with 2 for $i \neq j$ and $1 \leqslant t \leqslant n_k$. All the remaining edges are bi-coloured.

Finally Z^k will be a graph which does not contain a circuit of length less than or equal to l, and the chromatic number of which is k. The edges of Z^k are coloured with 2 the complement-edges with colour 1.

X^3:



Y^3:



The graphs considered above are special n-coloured complete graphs. If T^k denotes any one of $U^k, W^k, W_1^k, X^k, Y^k, Z^k$ then it has the property: if $k \to \infty$ then $\alpha(T^k) \to \infty$ so it follows that,

(*) $H = (H_1, H_2, \dots, H_n) \in \mathscr{H}$ involves that $H_i \subset T_i^k$ for some k in the i-th colour.

Now we turn to the proof of Theorem 2. Suppose that

$$H = (H_1, H_2, \dots, H_n) \notin \mathscr{A}(n) \cup Q^n \cup \mathscr{D}.$$

We prove that $H \notin \mathscr{H}$. We investigate two cases.

I. More than one H_i are empty graphs. Let $H_1, H_2, \dots, H_m$ be empty graphs $(m \geqslant 2)$ and $H_{m+1}, \dots, H_n$ be non-empty. Clearly $H_i \not\subset U^k$ for every k in the colour i, $(1 \leqslant i \leqslant n)$ so we conclude that $H \notin \mathscr{H}$ by (*).

II. We can choose H_1, H_2 so that H_i is non-empty for $3 \leqslant i \leqslant n$ and H_1, H_2 contains at least three vertices. A, B, C will denote the following graphs:

A:  B:  C: 

From $H \notin \mathscr{A}(n) \cup Q^n$ follows that at least one of the following six possibilities holds:

(i) $\bar{H}_1$ contains a circuit of length l and H_2 contains a triangle

(ii) $A \subset H_1$ and $A \subset H_2$

(iii) $B \subset H_1$ and $B \subset H_2$

(iv) $B \subset H_1$ and $A \subset H_2$

(v) $C \subset H_1$ and $A \subset H_2$

(vi) $C \subset H_1$ and $B \subset H_2$

(i) is impossible because $H_1, H_2 \not\subset Z^k$ in the first and second colour respectively $H_i \not\subset Z^k$ for $i \geqslant 3$ because H_i is non-empty.

Similar argument shows that the cases (ii) through (vi) are impossible. We can show that the graphs A, B, C which are subgraphs of H_1 and H_2 are not contained in our special graphs. In the cases (ii), (iii), (iv), (v), (vi) we use the graphs $U^k, W^k, Y^k, X^k, W_1^k$ respectively.

Proof of Theorem 3. Let $H = (H_1, H_2, \dots, H_n) \in \bar{Q}^n$ that is H_i can be written as the union of a_i points and b_i disjoint edges. Let $a = \max_i a_i$ and $b = (n-1) \max_i b_i + 1$. Q^* denotes a graph the complement of which consists of a disjoint vertices and b disjoint edges.

Clearly (cf. Theorem 1) $Q^* \in Q^n$. Let $G \in \mathscr{K}_s(n)$ for which $\alpha(G) > C_0(\underbrace{Q^*, Q^*, \ldots, Q^*}_{n \text{ times}})$. In this case G will contain Q^* in the i-th colour for some i. It is obvious (by the definition of a and b) that this subgraph contains H_j for some $j \neq i$.

Proof of theorem 4.

I. We prove $\mathscr{T}(n) \subset \mathscr{H}_s$ by induction on n. The case $n = 1$ is trivial. Assuming that $\mathscr{T}(n) \subset \mathscr{H}_s$ we prove that $\mathscr{T}(n+1) \subset \mathscr{H}_s$. Let

$H_1 = \ldots = H_n =$  and H_{n+1} be the complete k-partite graph which has k points in its classes. (It is clear that every k-partite complete graph is a subgraph of such a H_{n+1} for some k.)

Let $G \in \mathscr{K}_s(n+1)$ and

(i)

$$\alpha(G) > ((C_0(H_2, H_3, \ldots, H_n, H_{n+1}) - 1)(k[(k-1)(n-1) + 1] - 1) + k\,.$$

If $H_i \not\subset G$ in the i-th colour for $i = 1, 2, \ldots, n$ then G must be written as the union of disjoint complete graphs coloured by the i-th colour (the edges not belonging to these complete graphs are not coloured with colour i). Let us denote these complete graphs in colour 1 by

$A_1, A_2, \ldots, A_r$ and let $V(A_i) = \bigcup_{j=1}^{r_i} a^i_j$.

We can assume that $|V(A_i)| \geqslant |V(A_j)|$ for $i \geqslant j$. The number $|\{s\colon |A_s| \geqslant x\}|$ is denoted by t_x. We define B_u's as the "rows" of A_i's that is $B_u = \bigcup_{j=t+1}^{r} a^j_u$. We can write $V(G) = \bigcup_{i=1}^{t} A_i \cup \bigcup_u B_u$ and here the A_i's span complete graphs in colour 1 and the B_u's are n-coloured complete graphs so B_u can be covered by at most $C_0(H_2, H_3, \ldots, H_n, H_{n+1}) - 1$ complete one-coloured graphs by the inductive hypothesis. We get a covering of G by at most

(ii) $t_x + (C_0(H_2, \ldots, H_{n+1}) - 1)(x-1)$ complete graphs and comparing (i) and (ii) we conclude that

(iii) $t_x \geqslant k$ if $x = k[(k-1)(n-1)+1]$ that is for $1 \leqslant i \leqslant k$ $|V(A_i)| \geqslant k[(k-1)(n-1)+1]$. Let us choose the points $a_1^1, \ldots, a_1^k$ from $V(A_1)$. These points are connected with at most $k(n-1)$ points of $V(A_j)$, $(2 \leqslant j \leqslant k)$ in the colours $2, 3, \ldots, n$. We omit these points from $\bigcup_{i=2}^{k} V(A_i)$. Now we continue by choosing $a_{i_1}^2, a_{i_2}^2, \ldots, a_{i_k}^2$ from the reduced set $V(A_2)$ and remove from $\bigcup_{i=3}^{k} V(A_i)$ the points which are connected with a_{i_n} by edges of $2, 3, \ldots, n$ colour. The condition $|V(A_i)| \geqslant k[(k-1)(n-1)+1]$, $(1 \leqslant i \leqslant k)$ ensures that the process can be repeated until we have chosen k points from A_k. The graph spanned by the resulting vertex-set is isomorphic to H_{n+1} in the $n+1$-th colour.

II. We prove here that $\bar{\mathscr{T}}(n) \subset \mathscr{H}_s$.

(a) $\bar{\mathscr{T}}(2) \subset \mathscr{H}_s$ follows from $\mathscr{T}(2) \subset \mathscr{H}_s$ by symmetry.

(b) for $n > 2$ let $H_1 = H_2 = \ldots = H_{n-1} = \bar{T}$ and $H_n \in \bar{\mathscr{B}}_k$.

Let $G \in \mathscr{K}_s(n)$ for which

(iv) $\alpha(G) > C_0(H_1, H_n) + 1$.

We prove that in this case $H_i \subset G$ in the colour i for at least one i i.e. $H = (H_1, \ldots, H_n) \in \mathscr{H}_s$. If there is $A, B, C \in V(G)$ so that AB and AC edges have different colour from the colour-set $1, 2, \ldots, n-1$ then $H_i \subset G$ for some $i \leqslant n-1$. Otherwise $V(G) - P = X \cup Y$ for an arbitrary $P \in V(G)$ where the edges between P and X have colour i, $(i \leqslant n-1)$ and the edges between P and Y have colour n. Moreover the edges between X and Y have to be of colour i and the edges in Y have to be of colour n. The edges of X are coloured with colour i and n. We conclude that the set X spans a two-coloured complete graph and $P \cup Y$ spans a one-coloured complete graph. $\alpha(X) \geqslant C_0(H_1, H_n) + 1$ by condition (iv) so we can apply (a) for X which proves our statement.

The proof of Theorem 5.

(a) First we prove theorem for the case and when only one B is non-empty, that is

$$H_1 = A \cup B, \quad H_2, H_3, \ldots, H_n$$

are complete graphs.

Let $G \in \mathscr{K}_s(n)$ and

(i) $\alpha(G) > [R_0(A, H_2, \ldots, H_n)]^{n-1}$.

We assert that there is $P \in V(G)$ and a colour i for $2 \leqslant i \leqslant n$ so that at least $R_0(A, H_2, \ldots, H_n)$ edges starting from P have colour i.

Supposing the contrary, the graph G is considered as a one-coloured graph in the colour 2. Every vertex of G has degree of at most $R_0(A, H_2, \ldots, H_n) - 1$ so G is at most $R_0(A, H_2, \ldots, H_n)$-chromatic i.e. $V(G) = \bigcup_{j=1}^{t} A_j$ where $t \leqslant R_0(A, H_2, \ldots, H_n)$ and $A_t \in \mathscr{K}_s(n-1)$. Repeating this argument we see that G can be covered by at most $[R_0(A, H_2, \ldots, H_n)]^{n-1}$ complete graphs of colour 1 which contradicts to (i).

We can assume therefore the existence of $P \in V(G)$ and $X \subset V(G)$ such that the edges between P and X are coloured with i, $(2 \leqslant i \leqslant n)$ and $|X| \geqslant R_0(A, H_1, \ldots, H_n)$. Applying Ramsey's theorem, $A \subset X$ in colour 1 i.e. $P \cup A$ isomorphic to H_1 in colour 1 or at least one j, $(2 \leqslant j \leqslant n)$, $H_j \subset X$ in the j-th colour.

(b) Let $H = (H_1, H_2, \ldots, H_n) \in \mathscr{L}^n$ where $H_i = A_i \cup B_i$ and the B_i's are one-point graphs. We prove that the existence of $C_0(H_1, \ldots, H_n)$ follows from the existence of $C_0(H_1, A_2, \ldots, A_n) = t_1$, $C_0(A_1, H_2, A_3, \ldots, A_n), \ldots, C_0(A_1, \ldots, A_{n-2}, H_{n-1}, A_n) = t_{n-1}$, $C_0(A_1, \ldots, A_{n-1}, H_n) = t_n$ which was proved in (a). Let $G \in \mathscr{K}_s(n)$ and

(ii) $\alpha(G) > \sum_{i=1}^{n} t_i + 1$.

For any $P \in V(G)$ let

$$F_i = \{R \colon R \in V(G), \ RP \text{ edge has colour } i\} .$$

We have $\alpha(F_i) \geq t_i$ for at least one i (because of (ii)) so $H_i \subset F_i$ in the colour i or $A_j \subset F_i$ in the colour j for some $j \neq i$ and $P \cup A_j$ isomorphic to H_j. Therefore $H \in \mathscr{H}_s$ and the statement $\mathscr{L}^n(n) \subset \mathscr{H}_s$ is proved.

(c) $\bar{\mathscr{L}}^n(n) \subset \mathscr{H}_s$ is proved in the following way: let $H = (H_1, \ldots, H_n) \in \bar{\mathscr{L}}^n(n)$ and $H_i = A_i \cup B_i$. We define $H' = (H'_1, \ldots, H'_n)$ as follows: $H'_i = A' \cup B_i$ where $|V(A')| = \sum_{i=1}^{n} |V(A_i)|$ and A' is a complete graph. $H' \in \mathscr{L}^1(1)$ implies that $H' \in \mathscr{H}_s$ i.e. $C_0(H')$ exists. Let $G \in \mathscr{K}_s(n)$ for which

(iii) $\alpha(G) > C_0(H')$.

Condition (iii) implies that $H'_{i_0} \subset G$ in the colour i_0 for at least one i_0, that is every edge between A' and B_{i_0} has colour $1, 2, \ldots, i_0 - 1, i_0 + 1, \ldots, n$. The number of edges of this type is $\sum_{i=1}^{n} |V(A_i)|$ so we can choose for some $j \neq i_0$ $|V(A_j)|$ edges from them. The subgraph spanned by these edges is isomorphic to H_j so our statement follows.

Proof of Proposition 1. Let $H_1 = (H_1, \ldots, H_n) \in \mathscr{D}^*$ and suppose that H_1 and H_2 are empty graphs. Let $G \in \mathscr{K}_s(n)$ and H' be a complete graph so that $|V(H')| = \max(|V(H_1)|, |V(H_2)|)$. If $\alpha(G) > R_0(\underbrace{H', H', \ldots, H'}_{n \text{ times}})$ then G will contain H' in the colour i for at least one i. Because of $i \neq 1$ or $i \neq 2$ we have $H_1 \subset G$ or $H_2 \subset G$ in the colour 1 or 2 respectively.

Note that $\mathscr{D}^*$ makes $\mathscr{H}_s$ asymmetric.

Proof of Theorem 6. Let us suppose that $H = (H_1, H_2, \ldots, H_n) \in \mathscr{H}_s$. We can assume that $H_2, \ldots, H_n$ are not empty graphs (cf. Proposition 1). Let X_k be the complete k-partite graph containing k^2 evenly ditributed points and let Y_k^l be a k-chromatic graph in which every circuit has length $> l$, $(l \geq 3)$. We will consider X_k and Y_k^l as elements of $\mathscr{K}_s(n)$ where the edges of the graph have colour 1 and the edges of the complement have colour 2. $\alpha(X_k) \to \infty$ if $k \to \infty$ so $H \in \mathcal{H}_s$ implies that $H_1 \subset X_k$ or $\bar{H}_2 \subset X_k$ in colour 1 and 2 respectively for $k \geq k_0$ i.e.

(i) H_1 or $\bar{H}_2$ is a complete k-partite graph.

Similarly $\alpha(\bar{X}_k) \to \infty$ if $k \to \infty$ hence

(ii) H_1 or $\bar{H}_2$ is the complement of a complete k partite graph.

By the same argument $(\alpha(Y_k^l) \to \infty$ and $\alpha(\bar{Y}_k^l) \to \infty$ if $k \to \infty)$ we have

(iii) H_1 or $\bar{H}_2$ contains no circuit.

(iv) H_1 or $\bar{H}_2$ is a graph the complement of which contains no circuit.

Let us analize which possibilities hold for H_1:

(a) (i), (ii), (iii), (iv) hold for H_1 or none of these – that is in this case $(H_1, H_2) \in \mathcal{D}$.

(b) The following four possibilities

$$\left.\begin{array}{ll}\text{(i)} & \text{is true}\\ \text{(ii)} & \text{is true}\\ \text{(i)} & \text{is false}\\ \text{(ii)} & \text{is false}\end{array}\right\} \text{ for } H_1 \text{ implies that } (H_1, H_2) \in \mathcal{T}(2) \cup \bar{\mathcal{T}}(2)\,.$$

(c) If (i) and (iii) or (ii) and (iv) hold for H_1 then

$(H_1, H_2) \in \mathcal{L}^2 \cup \bar{\mathcal{L}}^2$.

(d) If (i) and (iv) or (ii) and (iii) hold for H_1 then

$(H_1, H_2) \in Q^2 \cup \bar{Q}^2$.

(e) All the remaining cases implies that $(H_1, H_2) \in \mathcal{A}(2) \cup \bar{\mathcal{A}}(2)$.

The cases (a) - (e) show that $H \dot{\in} \mathcal{H}_s - \mathcal{D}^*$ involves $(H_i, H_j) \in$ $\in \mathcal{D} \cup \mathcal{T}(2) \cup \bar{\mathcal{T}}(2) \cup \mathcal{L}^2 \cup \bar{\mathcal{L}}^2 \cup Q^2 \cup \bar{Q}^2 \cup \mathcal{A}(2) \cup \bar{\mathcal{A}}(2)$ for any

$1 \leqslant i,\ j \leqslant n$. From this the theorem easily follows.

Proof of Theorem 7. The upper bound follows from Theorem 8. The lower bound can be derived by examining a special k-chromatic graph without triangle, namely the Mycielski-graph. [4]

Proof of Theorem 8. We prove by induction on n.

(a) $n = 3$. Let $P \in V(G)$, we denote the set of points connected with P by A and $B = V(G) - (A \cup P)$. Let $B = \bigcup_{i=1}^{t} B_i$ where the B_i-s are the components of B. If for every B_i, $|V(B_i)| = 1$ then G would be 2-chromatic. Hence there is a B_{i_0} so that $|V(B_{i_0})| \geqslant 2$. The connectivity of G implies the existence of a $P \in A$ and an $R \in V(B_{i_0})$ so that $PR \in E(G)$. Finally we can choose an S from $V(B_{i_0})$ so that $(R, S) \in E(G)$. The path $[P, Q, R, S]$ has the required property.

(b) The induction is similar to step (a): let P be an arbitrary point in a $n+1$-chromatic connected graph which contains no triangle and A and B are defined as in part (a). Let B' be an n-chromatic component of B. (There exists such a component because G is $n+1$ chromatic). There is a point $Q \in V(A)$ so that Q is connected with some point of B' because G is connected. Let us consider the subgraph $Q \cup B'$ in G. It is n-chromatic at least, connected and contains no triangle – hence there is a path of $n+1$ points in it starting from Q by the inductive hypothesis. The edge PQ extends this path to length of $n+1$ which proves our statement.

Proof of Theorem 9. See [3].

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