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HOW TO ORIENT THE EDGES OF A GRAPH?

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I. INTRODUCTION

Let $G(V, E)$ be a finite undirected graph with vertex set V and edge set E. Multiple edges are allowed but loops are excluded. The present paper examines orientations of the edges (the orientation of an edge means a replacement of that undirected edge by a directed one) under which the resulting directed graph $\vec{G}$ has certain properties. These properties are: A, A and B, A and C where

A: the outdegrees of $\vec{G}$ lie between two bounds given in advance. (Different vertices can have different bounds.)

B: $\vec{G}$ is strongly connected.

C: $\vec{G}$ has a directed tree with a given root.

II. DEFINITIONS

$\rho(V')$ and $\delta(V')$ will denote the indegree and outdegree of the vertex subset V' in a directed graph. If V' contains only one vertex x we shall write $\rho(x)$ and $\delta(x)$ instead of $\rho(\{x\})$ and $\delta(\{x\})$. We shall often use the well-known inequality:

$$\rho(V_1) + \rho(V_2) \geqslant \rho(V_1 \cup V_2) + \rho(V_1 \cap V_2) \quad \text{for} \quad V_1, V_2 \subseteq V$$

which holds for δ as well. $d(x)$ denotes the degree of vertex x in an undirected graph.

$u(x)$ and $l(x)$ denote integer functions defined on the vertices of an undirected graph.

The shorter terms graph and digraph will be used throughout the paper instead of undirected and directed graph.

III. ORIENTATIONS WITH BOUNDS

Theorem 1. *A graph $G(V, E)$ has an orientation satisfying $\delta(x) \leqslant u(x)$ for $x \in V$ if and only if*

(1) $$|E_{V'}| \leqslant \sum_{x \in V'} u(x) \quad \textit{for all} \quad V' \subseteq V$$

where $E_{V'}$ denotes the set of edges in V'.

Proof. The necessity of (1) is obvious therefore we restrict ourselves to proving the sufficiency. The proof is an algorithm starting from an arbitrary orientation which may contain "wrong" vertices, that is vertices with $\delta(x) > u(x)$. At every step the algorithm yields a "better" orientation or a subset V' violating (1). An orientation is "better" if the sum of "surpluses" on the wrong vertices

$$\sum_{x \text{ is wrong}} (\delta(x) - u(x))$$

is smaller.

Let w be a wrong vertex: $\delta(w) > u(w)$. The set of vertices which can be reached from w by a directed path is denoted by V'. There are two cases:

A. There is a vertex $x \in V'$ for which $\delta(x) < u(x)$. In this case we choose a (directed) path from w to x and reverse the orientations along this path. Clearly we gain a "better" orientation because w is better, x is not wrong and the outdegrees of the other vertices on the path remained unchanged.

B. $\delta(x) \geqslant u(x)$ for every $x \in V'$. In this case $|E_{V'}| = \sum_{x \in V'} \delta(x)$ because no edge leaves V' but $\sum_{x \in V'} \delta(x) > \sum_{x \in V'} u(x)$ since $\delta(w) > u(w)$ and $\delta(x) \geqslant u(x)$ for any other vertex. The set V' therefore violates (1) and the proof is complete.∎

Theorem 2. *The graph $G(V, E)$ has an orientation satisfying $\delta(x) \geqslant \geqslant l(x)$ if and only if*

$$(2) \qquad |\widetilde{E}_{V'}| \geqslant \sum_{x \in V'} l(x) \quad \textit{for all} \quad V' \subseteq V$$

where $\widetilde{E}_{V'}$ denotes the set of edges incident to V'.

Proof. We can define the function $u(x)$ as $u(x) = d(x) - l(x)$ and we can apply Theorem 1. It is easy to see that for this $u(x)$ condition (1) is equivalent to (2) which proves the theorem.∎

Theorem 3. *The graph $G(V, E)$ has an orientation satisfying $l(x) \leqslant \leqslant \delta(x) \leqslant u(x)$ if and only if both* (1) *and* (2) *hold.* *($l(x) \leqslant u(x)$ for $x \in V$ is assumed.)*

Proof. (1) and (2) are obviously necessary. Let us start from an orientation for which $l(x) \leqslant \delta(x)$ for every $x \in V$. (There exists such an orientation by Theorem 2.) If we apply the algorithm given in the proof of Theorem 1 with the function $u(x)$ then, during the algorithm, the outdegree of a vertex x will decrease by one only if $\delta(x) > u(x)$ which means that $l(x) \leqslant \delta(x)$ holds after the algorithm.∎

Remark. The problems mentioned so far can be formulated as network-flow problems and our theorems follow from well-known results. We proved them in a direct way because they are suitable to prove our further results.

IV. STRONGLY CONNECTED ORIENTATIONS WITH BOUNDS

Definition. A digraph is strongly connected if for any two vertices x, y there is a directed path from x to y.

The following theorems are well-known:

Theorem. *A graph G has a strongly connected orientation if and only if G is bridgeless.*

Theorem. *The digraph G is strongly connected if and only if $\delta(V') > 0$ for any proper subset V' of V.*

Now we consider strongly connected orientations for a given graph satisfying upper and lower bound conditions for the outdegrees.

Theorem 4. *The bridgeless graph $G(V, E)$ has a strongly connected orientation satisfying $\delta(x) \leqslant u(x)$ if and only if*

$$|E_{V'}| + C_{G-V'} \leqslant \sum_{x \in V'} u(x) \quad \textit{for all} \quad V' \subseteq V \tag{4}$$

where $C_{G-V'}$ denotes the number of components in $G - V'$.

Proof. Let us assume that we have a required orientation of G and consider a subset V' of V. All the components of $G - V'$ can be reached from V' by a directed path, therefore at least $C_{G-V'}$ edges leave V'. It follows that at least $|E_{V'}| + C_{G-V'}$ edges have the origin in V'. On the other hand this number is at most $\sum_{x \in V'} u(x)$ from the condition so (4) is necessary.

The sufficiency of (4) is proved by an algorithm similar to the one used in the proof of Theorem 1. The algorithm starts from an arbitrary strongly connected orientation of G. If we are lucky it has no "wrong" vertices $(\delta(x) > u(x))$ and we can stop. Suppose that w is wrong. There exists a vertex x with $\delta(x) < u(x)$ because otherwise $V' = V$ violates (4). The strong connectivity guarantees at least one directed path from w to x. We distinguish two cases:

A. There are two edge-disjoint paths from w to x. Let us reverse the orientations along one of the paths. The new orientation is clearly strongly connected and is "better" because the sum of surpluses on the "wrong" vertices is decreased by one.

B. There are no two edge-disjoint paths from w to x. According to a variation of a well-known theorem of Menger [1] there is an $S_x \subset V$ for which

(i) $x \in S_x$, $w \notin S_x$ and $\rho(S_x) = 1$.

We choose S_x maximal with property (i).

Proposition 1. *Let* x, y *be vertices so that* $\delta(x) < u(x)$, $\delta(y) < u(y)$ *and construct* S_x *and* S_y *as shown before.* $S_x \cap S_y \neq \phi$ *implies* $S_x = S_y$.

Proof. Since

$$1 + 1 \leqslant \rho(S_x \cap S_y) + \rho(S_x \cup S_y) \leqslant \rho(S_x) + \rho(S_y) = 1 + 1$$

using the submodularity of ρ, therefore $\rho(S_x \cap S_y) + \rho(S_x \cup S_y) = 2$. $S_x \cap S_y$ and $S_x \cup S_y$ are non-empty so $\rho > 0$ on them, that is $\rho(S_x \cup S_y) = 1$ and hence $S_x \cup S_y$ satisfies (i). The maximality of S_x and S_y implies $S_x = S_x \cup S_y = S_y$ and the proposition is proved.∎

Proposition 2. S_x *and* S_y *is defined as before.* $S_x \neq S_y$ *implies that there are no edges between* S_x *and* S_y.

Proof. If ab is an edge from S_x to S_y then $\rho(S_x \cup S_y) = \rho(S_x) = 1$ which contradicts to the maximality of S_x.∎

Let us define $V' = V - \bigcup_{\delta(x) < u(x)} S_x$. $V' \neq \phi$ because $w \in V'$. We show that V' violates (4). The two propositions imply that the components of $G - V'$ are the distinct sets S_x's. By the definition of S_x there is exactly one edge from V' to every S_x therefore exactly $|E_{V'}| + C_{G-V'}$ edges have the origin in V', that is

$$|E_{V'}| + C_{G-V'} = \sum_{x \in V'} \delta(x).$$

The sum on the right side is greater than $\sum_{x \in V'} u(x)$ because the vertices for which $\delta(x) < u(x)$ are covered by the S_x's and $\delta(w) > u(w)$.

We conclude that $|E_{V'}| + C_{G-V'} > \sum_{x \in V'} u(x)$ that is V' violates condition (4).∎

Remark. The method used in the proof of Theorem 4 is really an algorithm since the sets S_x can be constructed easily. The Ford – Fulkerson labelling algorithm either assures the two edge-disjoint paths from w to x or gives a maximal S_x. We present here the algorithm in flowchart form (see Fig. 1) and note that all proofs of this paper can be presented in such a form.

Theorem 5. *A bridgeless graph $G(V, E)$ has an orientation satisfying $\delta(x) \geqslant l(x)$ for any $x \in V$ if and only if*

$$(5) \qquad |\widetilde{E}_{V'}| - C_{G-V'} \geqslant \sum_{x \in V'} l(x) \quad \text{for all} \quad V' \subseteq V$$

where $\widetilde{E}_{V'}$ denotes the set of edges incident to V'.

Proof. The details are left to the reader. The theorem is in duality with Theorem 4 and is based on the fact that the existence of the required orientation is equivalent to the existence of a strongly connected orientation where $\delta(x) \leqslant d(x) - l(x)$. Choosing $u(x) = d(x) - l(x)$ and applying Theorem 4 we can prove Theorem 5.∎

Theorem 6. *A bridgeless graph $G(V, E)$ has an orientation satisfying $l(x) \leqslant \delta(x) \leqslant u(x)$ if and only if* (4) *and* (5) *hold simultaneously.* *($l(x) \leqslant$ $\leqslant u(x)$ is assumed for all $x \in V$.)*

Proof. The conditions are obviously necessary. For the sufficiency we consider a strongly connected orientation satisfying $l(x) \leqslant \delta(x)$ for all $x \in V$. (Theorem 5 guarantees such an orientation.) Applying the algorithm of Theorem 4 we can reach the required orientation or a subset $V' \subseteq V$ violating (4).∎

V. ORIENTATIONS WITH A ROOTED TREE AND BOUNDS

In this section the strong connectivity is replaced by a weaker condition: we try to find an orientation so that every vertex can be reached by a directed path from a specified vertex r (we shall say that the digraph

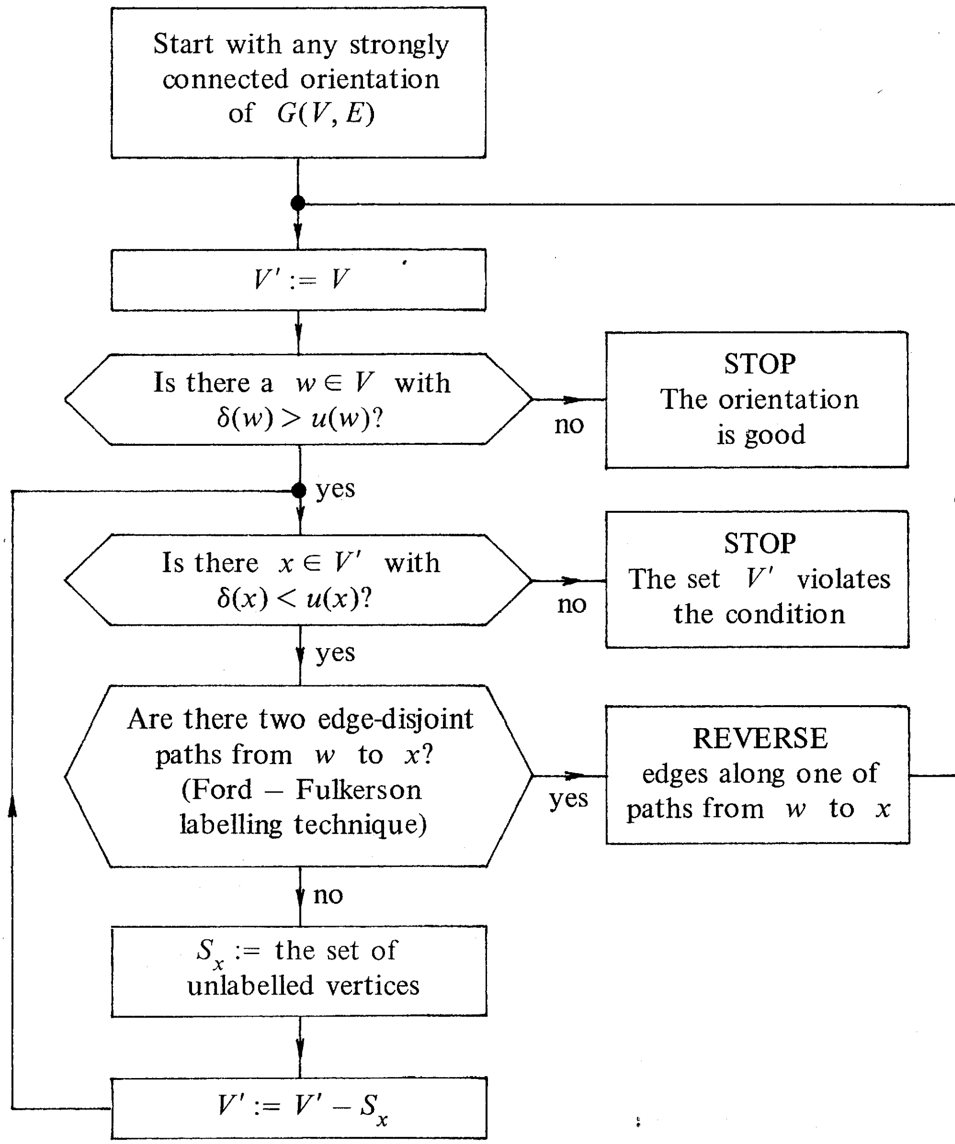
Start with any strongly connected orientation of $G(V, E)$
$V' := V$
Is there a $w \in V$ with $\delta(w) > u(w)$?
no
STOP
The orientation is good
yes
Is there $x \in V'$ with $\delta(x) < u(x)$?
no
STOP
The set V' violates the condition
yes
Are there two edge-disjoint paths from w to x? (Ford – Fulkerson labelling technique)
yes
REVERSE
edges along one of paths from w to x
no
S_x := the set of unlabelled vertices
$V' := V' - S_x$

Fig. 1

has an r-tree), moreover the outdegrees lie between bounds given in advance.

Theorem 7. *Let $G(V,E)$ be a graph with a given vertex r and $u(x)$ is an integer function on V. $G(V,E)$ has an orientation with an r-tree and satisfies $\delta(x) \leqslant u(x)$ if and only if*

$$(7) \qquad |E_{V'}| + \bar{C}_{G-V'} \leqslant \sum_{x \in V'} u(x) \quad \text{for all} \quad V' \subseteq V$$

where $\bar{C}_{G-V'}$ is the number of components of $G-V'$ which do not contain r. ($\bar{C}_{G-V'} = C_{G-V'}$ if $r \in V'$ and $\bar{C}_{G-V'} = C_{G-V'} - 1$ if $r \notin V'$.)

Remark. In the theory of programming the notion of the flow-graph is well-known. A flow-graph is a directed graph which has an r-tree. In that case r represents the starting point of the computation. An important role is given to the flow-graphs where the outdegrees are at most 2. The problem of characterizing the "skeleton" of such a flow-graph arose. Theorem 7 gives the answer if we choose $u(x) \equiv 2$.

Proof. We prove the sufficiency – the proof of necessity is left to the reader.

Choosing $V' = \phi$, condition (7) states that our graph is connected. We can orient it therefore to contain an r-tree. The algorithm starts from that orientation and tries to improve it i.e. to decrease $\sum(\delta(x) - u(x))$ on the "wrong" vertices – and not to loose the property of having an r-tree.

Let w be a wrong vertex $(\delta(w) > u(w))$ and V_1 be the set of vertices which can be reached from w by a directed path. Two cases are distinguished:

A. There is an $x \in V_1$ with $\delta(x) < u(x)$ reachable with two edge-disjoint paths, one of them from w and the other is from r. In this case we reverse the orientation along the path wx and we have a "better" orientation.

B. If A is not true then by a version of Menger's theorem, and using the fact that x can be reached from w and r, every $x \in V_1$ for which $\delta(x) < u(x)$ is contained in a set $S_x \subset V$ with the properties:

(ii) $x \in S_x$; $r, w \notin S_x$; $\rho(S_x) = 1$.

Let us choose the sets S_x's to be maximal with property (ii). The following two propositions are analogous to those of used in the proof of Theorem 4 and their proofs are quite the same.

Proposition 3. $S_x \cap S_y \neq \phi$ *implies* $S_x = S_y$.

Proposition 4. $S_x \neq S_y$ *implies that there are no edges between* S_x *and* S_y.

We define $V_2 = V - V_1$.

Proposition 5. $V_2 \cap S_x = \phi$ *for any* S_x.

Proof. Assume that $V_2 \cap S_x \neq \phi$. In this case clearly $V_2 \neq \phi$ and $r \in V_2$.

$$0 + 1 = \rho(V_2) + \rho(S_x) \geqslant \rho(V_2 \cup S_x) + \rho(V_2 \cap S_x) \geqslant 1 + 1$$

and it is a contradiction.∎

Proposition 6. *There are no edges between* V_2 *and* S_x.

Proof. It is clear that there are no edges from V_1 to V_2. We know from (ii) that $\rho(S_x) = 1$ but the unique edge entering S_x is on the path wx which lies in V_1 – therefore no edge can enter S_x from V_2.∎

Now we define $V' = V_1 - \bigcup\limits_{\substack{x \in V_1 \\ \delta(x) < u(x)}} S_x$.

Propositions 3-6 guarantee that the components of $G - V'$ are exactly the different S_x's and V_2 if $V_2 \neq \phi$. ($V_2 \neq \phi$ implies $r \in V_2$). The number of edges, therefore, with origin in V' is $|E_{V'}| + \bar{C}_{G-V'} = \sum\limits_{x \in V'} \delta(x)$ but the right side is greater than $\sum\limits_{x \in V'} u(x)$ because $\delta(x) \geqslant u(x)$ on V' and $\delta(w) > u(w)$, so $|E_{V'}| + \bar{C}_{G-V'} > \sum\limits_{x \in V'} u(x)$ which contradicts (7).

Theorem 8. *A connected graph* $G(V,E)$ *has an orientation with an r-tree and satisfying* $\delta(x) \geqslant l(x)$ *if and only if*

$$(8)\qquad \begin{aligned} |\widetilde{E}_{V'}| &\geqslant \sum_{x \in V'} l(x) \quad \text{if} \quad r \in V' \subseteq V \\ |\widetilde{E}_{V'}| - 1 &\geqslant \sum_{x \in V'} l(x) \quad \text{if} \quad r \notin V' \subseteq V \end{aligned}$$

where $\widetilde{E}_{V'}$ *denotes the edge-set incident to* V'.

Proof. We prove only the sufficiency. The proof proceeds on the same line as before. Starting from an orientation with an r-tree, we improve on it or we find a V' violating (8). Let w be a "wrong" vertex i.e. $\delta(w) < l(w)$ and the set of vertices from which w is reachable by a directed path is denoted by V_1. Clearly $r \in V_1$. If $\delta(x) \leqslant l(x)$ for every $x \in V_1$ then V_1 violates the first part of (8). Clearly $\rho(V_1) = 0$ and therefore

$$|\widetilde{E}_{V_1}| = \sum_{x \in V_1} \delta(x) < \sum_{x \in V_1} l(x).$$

We can assume that there is at least one $x \in V_1$ with $\delta(x) > l(x)$. We have two cases:

A. There are two edge-disjoint paths terminating at w and starting from x and r respectively. We can reverse the edges along the path xw and we get a better orientation without destroying the r-tree property.

B. If **A** is false then a variation of Menger's theorem guarantees an S_x for any $x \in V_1$ with $\delta(x) > l(x)$ with the property:

(iii) $r, x \in S_x$, $w \notin S_x$ and $\delta(S_x) = 1$.

Choose S_x to be maximal with property (iii).

Proposition 7. $y \in V_1 - S_x$ *implies* $\delta(y) \leqslant l(y)$.

Proof. If $\delta(y) > l(y)$ for a $y \in V_1 - S_x$ then S_y is defined and $1 + 1 = \delta(S_x) + \delta(S_y) \geqslant \delta(S_x \cup S_y) + \delta(S_x \cap S_y) \geqslant 1 + 1$ which means that $S_x \cup S_y$ satisfies (iii) which contradicts to the maximality of S_x. ∎

Proposition 8. $S_x \supset V - V_1$.

Proof. $\delta(V - V_1) = 0$ hence $S_x \cup (V - V_1)$ satisfies (iii).∎

We prove that $V' = V_1 - S_x$ violates the second part of condition (8). Clearly $r \notin V'$. There is exactly one edge into V' from S_x, therefore $|\widetilde{E}_{V'}| - 1 = \sum_{x \in V'} \delta(x)$ but the right side is less than $\sum_{x \in V'} l(x)$ because of Propositions 7 and 8 and $\delta(w) < l(w)$ so V' really violates (8).∎

Remark. It is interesting to note that Theorem 8 does not follow from Theorem 7 by "duality" as was the case for Theorem 2 and Theorem 5.∎

We do not think Theorem 9 will surprise anyone:

Theorem 9. *A graph $G(V, E)$ has an orientation with an r-tree satisfying $l(x) \leqslant \delta(x) \leqslant u(x)$ if and only if* (7) *and* (8) *hold simultaneously.*

Proof. Same as before.∎

VI. A GENERAL PROBLEM FOR SUBSET OUTDEGREES

In this last section we formulate a generalization of the problems discussed so far. Let $l(X)$ be an integer function on the subsets of V, where $G(V, E)$ is an undirected graph.

Problem. What conditions should be imposed on G in order to possess an orientation under which

$$\delta(X) \geqslant l(X) \qquad \text{for any} \qquad X \subseteq V?$$

It is easy to see that all problems of this paper are special cases of this general problem. The strong connectivity, for example, is equivalent to $\delta(X) \geqslant 1$ for all $X \subset V$.

The general problem is not solved but there are numerous interesting special cases. A theorem of Nash-Williams [2] answers the case $l(X) \equiv k$.

A forthcoming paper [3] of one of the authors (A. Frank) will answer some further special cases e.g. when the vertices have lower and upper bounds and a k-connected orientation is needed. Another special case is when $l(X)$ is supermodular.

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