On \((\delta, \chi)\)-bounded families of graphs

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May 28, 2010

Abstract

A family \(F\) of graphs is said to be \((\delta, \chi)\)-bounded if there exists a function \(f(x)\) satisfying \(f(x) \to \infty\) as \(x \to \infty\), such that for any graph \(G\) from the family, one has \(f(\delta(G)) \leq \chi(G)\), where \(\delta(G)\) and \(\chi(G)\) denotes the minimum degree and chromatic number of \(G\), respectively. Also for any set \(\{H_1, H_2, \ldots, H_k\}\) of graphs by \(\text{Forb}(H_1, H_2, \ldots, H_k)\) we mean the class of graphs that contain no \(H_i\) as an induced subgraph for any \(i = 1, \ldots, k\). In this paper we first answer affirmatively the question raised by the second author by showing that for any tree \(T\) and positive integer \(\ell\), \(\text{Forb}(T, K_{\ell, \ell})\) is a \((\delta, \chi)\)-bounded family. Then we obtain a necessary and sufficient condition for \(\text{Forb}(H_1, H_2, \ldots, H_k)\) to be a \((\delta, \chi)\)-bounded family, where \(\{H_1, H_2, \ldots, H_k\}\) is any given set of graphs. Next we study \((\delta, \chi)\)-boundedness of \(\text{Forb}(C)\) where \(C\) is an infinite collection of graphs. We show that for any positive integer \(\ell\), \(\text{Forb}(K_{\ell, \ell}, C_6, C_8, \ldots)\) is \((\delta, \chi)\)-bounded. Finally we show a similar result when \(C\) is a collection consisting of unicyclic graphs.

1 Introduction

A family \(F\) of graphs is said to be \((\delta, \chi)\)-bounded if there exists a function \(f(x)\) satisfying \(f(x) \to \infty\) as \(x \to \infty\), such that for any graph \(G\) from the family one has \(f(\delta(G)) \leq \chi(G)\), where \(\delta(G)\) and \(\chi(G)\) denotes the minimum degree and chromatic number of \(G\), respectively. Equivalently, the family \(F\) is \((\delta, \chi)\)-bounded if \(\delta(G_n) \to \infty\) implies \(\chi(G_n) \to \infty\).

\*Research supported in part by OTKA Grant No. K68322.
∞ for any sequence \( G_1, G_2, \ldots \) with \( G_n \in \mathcal{F} \). Motivated by Problem 4.3 in [6], the second author introduced and studied \((\delta, \chi)\)-bounded families of graphs (under the name of \(\delta\)-bounded families) in [10]. The so-called color-bound family of graphs mentioned in the related problem of [6] is a family for which there exists a function \( f(x) \) satisfying \( f(x) \to \infty \) as \( x \to \infty \), such that for any graph \( G \) from the family one has \( f(\text{col}(G)) \leq \chi(G) \), where \( \text{col}(G) \) is defined as \( \text{col}(G) = \max\{\delta(H) : H \subseteq G\} + 1 \). As shown in [10] if we restrict ourselves to hereditary (i.e. closed under taking induced subgraph) families then two concepts \((\delta, \chi)\)-bounded and color-bound are equivalent. The first specific results concerning \((\delta, \chi)\)-bounded families appeared in [10] where the following theorem was proved (in a somewhat different but equivalent form).

**Theorem 1 ([10])** For any set \( \mathcal{C} \) of graphs, \( \text{Forb}(\mathcal{C}) \) is \((\delta, \chi)\)-bounded if and only if there exists a constant \( c = c(\mathcal{C}) \) such that for any bipartite graph \( H \in \text{Forb}(\mathcal{C}) \) one has \( \delta(H) \leq c \).

Theorem 1 shows that to decide whether \( \text{Forb}(\mathcal{C}) \) is \((\delta, \chi)\)-bounded we may restrict ourselves to bipartite graphs. We shall make use of this result in proving the following theorems.

Similar to the concept of \((\delta, \chi)\)-bounded families is the concept of \(\chi\)-bounded families. A family \( \mathcal{F} \) of graphs is called \(\chi\)-bounded if for any sequence \( G_i \in \mathcal{F} \) such that \( \chi(G_i) \to \infty \), it follows that \( \omega(G_i) \to \infty \). The first author conjectured [2] (independently by Sumner [9]) the following

**Conjecture 1** For any fixed tree \( T \), \( \text{Forb}(T) \) is \(\chi\)-bounded.

## 2 Finite \((\delta, \chi)\)-bounded families

The first result in this section shows that for any tree \( T \) and positive integer \( \ell \), \( \text{Forb}(T, K_{\ell,\ell}) \) is \((\delta, \chi)\)-bounded which answers affirmatively a problem of [10].

**Theorem 2** For every fixed tree \( T \) and fixed integer \( \ell \), and any sequence \( G_i \in \text{Forb}(T, K_{\ell,\ell}) \), \( \delta(G_i) \to \infty \) implies \( \chi(G_i) \to \infty \).

We shall prove Theorem 2 in the following quantified form.

**Theorem 3** For every tree \( T \) and for positive integers \( \ell, k \) there exist a function \( f(T, \ell, k) \) with the following property. If \( G \) is a graph with \( \delta(G) \geq f(T, \ell, k) \) and \( \chi(G) \leq k \) then \( G \) contains either \( T \) or \( K_{\ell,\ell} \) as an induced subgraph.
In Theorem 3 we may assume that the tree $T$ is a complete $p$-ary tree of height $r$, $T^*_p$, because these trees contain any tree. Using Theorem 1 we note that to prove Theorem 3 it is enough to show the following lemma.

**Lemma 1** For every $p, r, l$ there exists $g(p, r, l)$ such that the following is true. Every bipartite graph $H$ with $\delta(H) \geq g(p, r, l)$ contains either $T^*_p$ or $K_{l, l}$ as an induced subgraph.

**Proof.** To prove the lemma, we prove slightly more. Call a subtree $T \subseteq H$ a distance tree rooted at $v \in V(H)$ if $T$ is rooted at $v$ and for every $w \in V(T)$ the distance of $v$ and $w$ in $T$ is the same as the distance of $v$ and $w$ in $H$. In other words, in a distance tree $T$, level $i$ of $T$, $L_i$, is a subset of the vertices at distance $i$ from $v$ in $H$. Notice that - a distance tree of $H$ is an induced subtree of $H$ if and only if $xy \in E(H)$ such that $x \in L_i$, $y \in L_{i+1}$ implies $xy \in E(T)$ (observe that in this statement it is important that $H$ is a bipartite graph otherwise $xy \in E(H)$ would be possible with $x, y \in L_i$).

We claim that with a suitable $g(p, r, l)$ lower bound for $\delta(H)$ every vertex of a bipartite graph $H$ is the root of an induced distance tree $T^*_p$ in $H$.

The claim is proved by induction on $r$. For $r = 1$, $g(p, 1, \ell) = p$ is a suitable function for every $\ell, p$. Assuming that $g(p, r, \ell)$ is defined for some $r \geq 1$ and for all $p, \ell$, define $P = p^{r+1}(\ell - 1)$ and

$$u = g(p, r + 1, \ell) = \max\{g(P, r, \ell), 1 + 2^{Pr\ell}(\max\{p - 1, \ell - 1\})\} \tag{1}$$

Suppose that $\delta(H) \geq u, v \in V(H)$. By induction, using that $u \geq g(P, r, \ell)$ by (1), we can find an induced distance tree $T = T^*_p$ rooted at $v$. In fact we shall only extend a subtree $T^*$ of $T$, defined as follows. Keep $p$ from the $P$ subtrees under the root and repeat this at each vertex of the levels $1, 2, \ldots, r - 2$. Finally, at level $r - 1$, keep all of the $P$ children at each vertex. (Here one can refine the proof to get better bounds.) Let $L$ denote the set of vertices of $T^*$ at level $r$, $L = \cup_{i=1}^{|A|} A_i$, where the vertices of $A_i$ have the same parent in $T^*$, $|A| = P$. Let $X \subseteq V(H) \setminus V(T^*)$ denote the set of vertices adjacent to some vertex of $L$. (In fact, since $T$ is a distance tree and $H$ is bipartite, $X \subseteq V(H) \setminus V(T^*)$.) Put the vertices of $X$ into equivalence classes, $x \equiv y$ if and only if $x, y$ are adjacent to the same subset of $L$. There are less than $q = 2^{Pr\ell}$ equivalence classes. Delete from $X$ all vertices of those equivalence classes that are adjacent to at least $\ell$ vertices of $L$. Since $H$ has no $K_{l, l}$ subgraph, at most $q(l - 1)$ vertices are deleted. Delete also from $X$ all vertices of those equivalence classes that have at most $p - 1$ vertices. During these deletions less than $q(\max\{p - 1, \ell - 1\}) < u - 1$ vertices were deleted, the set of remaining vertices is $Y$. It follows from (1) that every vertex of $L$ is adjacent to at least one vertex $y \in Y$ - in fact to at least $p$ vertices of $Y$ in the equivalence class of $y$.

Now we plan selecting vertex $x_i \in A_i$ so that each of them has a set $B_i$ of $p$ neighbors in $Y$, the $B_i$-s are pairwise disjoint and no $x_i$ is adjacent to any vertex in $B_j$ if $j \neq i$. Thus $\cup_{i=1}^r B_i$ extends $T^*$ to the required induced distance tree $T^*_{r+1}$.
Start with an arbitrary vertex $x_1 \in A_1$. There are at least $p$ neighbors of $x_1$ in an equivalence class of $Y$, define $B_1$ as $p$ of them. At most $\ell - 1$ vertices of $L$ define this class, thus we can select $x_2 \in A_2$ from a different from those. Now take any neighbor of $x_2$ and repeat the procedure by selecting $x_3 \in A_3$ different from the at most $2(\ell - 1)$ vertices that may define the previous classes. Since $|A_p| = P > (p^{r - 1})(\ell - 1)$, all these steps can be taken.

Using Theorem 2 we can characterize $(\delta, \chi)$-bounded families of the form $Forb(H_1, \ldots, H_k)$ where $\{H_1, \ldots, H_k\}$ is any finite set of graphs. In the following result by a star tree we mean any tree isomorphic to $K_{1,t}$ for some $t \geq 1$.

**Corollary 1** Given a finite set of graphs $\{H_1, H_2, \ldots, H_k\}$. Then $Forb(H_1, H_2, \ldots, H_k)$ is $(\delta, \chi)$-bounded if and only if one of the following holds:

(i) For some $i$, $H_i$ is a star tree.

(ii) For some $i$, $H_i$ is a forest and for some $j \neq i$, $H_i$ is complete bipartite graph.

**Proof.** Set for simplicity $\mathcal{F} = Forb(H_1, H_2, \ldots, H_k)$. First assume that $\mathcal{F}$ is $(\delta, \chi)$-bounded. From the well-known fact that for any $d$ and $g$ there are bipartite graphs of minimum degree $d$ and girth $g$, we obtain that some $H_i$ should be forest. If $H_i$ is star tree then (i) holds. Assume on contrary that none of $H_i$’s is neither star tree nor complete bipartite graph. Then $K_{n,n}$ belongs to $\mathcal{F}$ for some $n$. But this violates the assumption that $\mathcal{F}$ is $(\delta, \chi)$-bounded.

To prove the converse, first note that by a well known fact (see [10]) if $H_i$ is a star tree then $Forb(H_i)$ is $(\delta, \chi)$-bounded. Now since $\mathcal{F} \subseteq Forb(H_i)$ then $\mathcal{F}$ too is $(\delta, \chi)$-bounded. Now let (ii) hold. We may assume that $H_{i_0}$ is forest and $H_{j_0}$ is an induced subgraph of $K_{l,\ell}$ for some $l$. It is enough to show that $Forb(H_{i_0}, K_{l,\ell})$ is $(\delta, \chi)$-bounded. If $H_{i_0}$ is a tree then the assertion follows by Theorem [2]. Let $T_1, \ldots, T_k$ be the connected components of $H_{i_0}$ where $k \geq 2$. We add a new vertex $v$ and connect $v$ to each $T_i$ by an edge. The resulting graph is a tree denoted by $T$. We have $Forb(H_{i_0}, K_{l,\ell}) \subseteq Forb(T, K_{l,\ell})$ since $H_{i_0}$ is induced subgraph of $T$. The proof now completes by applying Theorem [2] for $Forb(T, K_{l,\ell})$. □

**3 Infinite $(\delta, \chi)$-bounded families**

In the sequel we consider $Forb(H_1, H_2, \ldots)$ where $\{H_1, H_2, \ldots\}$ is any infinite collection of graphs. When at least one of the $H_i$'s is tree then the related characterization problem is easy. The following corollary is immediate.

**Corollary 2** Let $T$ be any non star tree. Then $Forb(T, H_1, \ldots)$ is $(\delta, \chi)$-bounded if and only if at least one of $H_i$'s is complete bipartite graph.
When no graph is acyclic in our infinite collection $H_1, H_2, \ldots$ we face with non-trivial problems. The first result in this regard is a result from [8]. They showed that if $G$ is any even-cycle-free graph then $\text{col}(G) \leq 2\chi(G) + 1$. This shows that $\text{Forb}(C_4, C_6, C_8, \ldots)$ is $(\delta, \chi)$-bounded. Another result concerning even-cycles was obtained in [10] where the following theorem has been proved. Note that $d(G)$ stands for the average degree of $G$.

**Theorem 4** ([10]) Let $F$ be any set of even integers, $G$ a graph with $F \subseteq F(G)$ and $A = E \setminus F$ where $E$ is the set of even integers greater than two. Assume that $A = \{g_1, g_2, \ldots\}$.

Set $\lambda = 2d(d + 1)$ where $d = \gcd(g_1 - 2, g_2 - 2, \ldots)$. If $d \geq 4$ then

$$\chi(G) \geq \frac{d(G)}{\lambda} + 1.$$ 

In the sequel using a result from [4] we show that for any positive integer $\ell$, $\text{Forb}(K_{\ell,\ell}, C_6, C_8, C_{10}, \ldots)$ is $(\delta, \chi)$-bounded. For this purpose we need to introduce bipartite chordal graphs. A bipartite graph $H$ is said to be bipartite chordal if any cycle of length at least 6 in $H$ has at least one chord. Let $H$ be a bipartite graph with bipartition $(X, Y)$. A vertex $v$ of $H$ is simple if for any $u, u' \in N(v)$ either $N(u) \subset N(u')$ or $N(u') \subset N(u)$.

Suppose that $\mathcal{L} : v_1, v_2, \ldots, v_n$ is a vertex ordering of $H$. For each $i \geq 1$ denote $H[v_i, v_{i+1}, \ldots, v_n]$ by $H_i$. An ordering $\mathcal{L}$ is said to be a simple elimination ordering of $H$ if $v_i$ is a simple vertex in $H_i$ for each $i$. The following theorem first appeared in [4] (see also [5]).

**Theorem 5** ([4]) Let $H$ be a bipartite graph with bipartition $(X, Y)$. Then $H$ is chordal bipartite if and only if it has a simple elimination ordering. Furthermore, suppose that $H$ is chordal bipartite. Then there is a simple ordering $y_1, \ldots, y_m, x_1, \ldots, x_n$ where $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$, such that if $x_i$ and $x_k$ with $i < k$ are both neighbors of some $y_j$, then $N_{H'}(x_i) \subseteq N_{H'}(x_k)$ where $H'$ is the subgraph of $H$ induced by $\{y_j, \ldots, y_m, x_1, \ldots, x_n\}$.

**Theorem 6** $\text{Forb}(K_{\ell,\ell}, C_6, C_8, C_{10}, \ldots)$ is $(\delta, \chi)$-bounded.

**Proof.** By Theorem 1 it is enough to show that the minimum degree of any bipartite graph $H \in \text{Forb}(K_{\ell,\ell}, C_6, C_8, C_{10}, \ldots)$ is at most $\ell - 1$.

Let $H$ be a bipartite $(K_{\ell,\ell}, C_6, C_8, C_{10}, \ldots)$-free graph with $\delta(H) \geq \ell$. Let $y_1, \ldots, y_m, x_1, \ldots, x_n$ be the simple ordering guaranteed by Theorem 5. The vertex $y_1$ has at least $k$ neighbors say $z_1, \ldots, z_k$ such that $N(z_1) \subseteq N(z_2) \subseteq \ldots \subseteq N(z_k)$. Now since $d_Y(z_1) \geq k$ so there are $k$ vertices in $Y$ which are all adjacent to $z_1$. From other side $N(z_1) \subseteq N(z_i)$ for any $i = 1, \ldots, k$. Therefore all these $k$ neighbors of $z_1$ are also adjacent to $z_i$ for any $i$. This introduces a subgraph of $H$ isomorphic to $K_{\ell,\ell}$, a contradiction. \qed
We conclude this section with another \((\delta, \chi)\)-bounded (infinite) family of graphs. By a unicyclic graph \(G\) we mean any connected graph which contains only one cycle. Such a graph is either a cycle or consists of an induced cycle \(C\) of length say \(i\) and a number of at most \(i\) induced subtrees such that each one intersects \(C\) in exactly one vertex. We call these subtrees (which intersects \(C\) in exactly one vertex) the attaching subtrees of \(G\). Recall from the previous section that \(T^r_p\) is the \(p\)-ary tree of height \(r\). For any positive integers \(p\) and \(r\) by a \((p, r)\)-unicyclic graph we mean any unicyclic graph whose attaching subtrees are subgraph of \(T^r_p\). We also need to introduce some special instances of unicyclic graphs. For any positive integers \(p\), \(r\) and even integer \(i\), let us denote the graph consisting of the even cycle \(C\) of length \(i\) and \(i\) vertex disjoint copies of \(T^r_p\) which are attached to the cycle \(C\) by \(U_{i, p, r}\) (to each vertex of \(C\) one copy of \(T^r_p\) is attached).

**Proposition 1** For any positive integers \(t\), \(p\) and \(r\), there exists a constant \(c = c(t, p, r)\) such that for any \(K_{2, t}\)-free bipartite graph \(H\) if \(\delta(H) \geq c\) then for some even integer \(i\), \(H\) contains an induced subgraph isomorphic to \(U_{i, p, r}\).

**Proof.** Let \(H\) be any \(K_{2, t}\)-free bipartite graph. There are two possibilities for the girth \(g(H)\) of \(H\).

**Case 1.** \(g(H) \geq 4r + 3\). Let \(C\) be any smallest cycle in \(H\). Since \(H\) is bipartite then \(C\) has an even length say \(i = g(H)\). We prove by induction on \(k\) with \(0 \leq k \leq i\) that if \(\delta(H) \geq g(p, r, t) + 2\) then \(H\) contains an induced subgraph isomorphic to the graph obtained by \(C\) and \(k\) attached copies of \(T^r_p\), where \(g(p, r, t)\) is as in Lemma 1. The assertion is trivial for \(k = 0\). Assume that it is true for \(k\) and we prove it for \(k + 1\). By induction hypothesis we may assume that \(H\) contains an induced subgraph \(L\) consisting of the cycle \(C\) plus \(k\) copies of \(T^r_p\) attached to \(C\). Let \(v\) be a vertex of \(C\) at which no tree is attached. Let \(e\) and \(e'\) be two edges on \(C\) which are incident with the vertex \(v\). We apply Lemma 1 for \(H \setminus \{e, e'\}\). Note that since \(\delta(H) \geq g(p, r, t) + 2\) then the degree of \(v\) in \(H \setminus \{e, e'\}\) is at least \(g(p, r, t)\). We find an induced copy of \(T^r_p\) grown from \(v\) in \(H \setminus \{e, e'\}\). Denote this copy of \(T^r_p\) by \(T_0\). Consider the union graph \(L \cup T_0\). We show that \(L \cup T_0\) is induced in \(H\). We only need to show that no vertex of \(T_0\) is adjacent to any vertex of \(L\). The distance of any vertex in \(T_0\) from the farthest vertex in \(C\) is at most \(r + i/2\). The distance of any vertex in the previous copies of \(T^r_p\) in \(L\) from \(C\) is at most \(r\). Then any two vertices in \(T_0 \cup L\) have distance at most \(2r + i/2\). Now if there exists an edge between two such vertices we obtain a cycle of length at most \(2r + i/2 + 1\) in \(H\). By our condition on the girth of \(H\) we obtain \(2r + i/2 + 1 < g(H)\), a contradiction. This proves our induction assertion for \(k + 1\), in particular the assertion is true for \(k = i\). But this means that \(H\) contains the cycle \(C\) with \(i\) copies of \(T^r_p\) attached to \(C\) in induced form. The latter subgraph is \(U_{i, p, r}\). This completes the proof in this case.
Case 2. \( g(H) \leq 4r + 2 \). In this case we prove a stronger claim as follows. If \( H \) is any \( K_{2,t} \)-free bipartite graph and \( \delta(H) \geq (4r + 2)(t - 1)(\max\{r + 1, p^{r+1}\}) + 1 \) with \( g(H) = i \) then \( H \) contains any graph \( G \) which is obtained by attaching \( k \) trees \( T_1, \ldots , T_k \) to the cycle of length \( i \) such that any \( T_j \) is a subtree of \( T'_p \) and \( k \) is any integer with \( 0 \leq k \leq i \). It is clear that if we prove this claim then the main assertion is also proved.

Now let \( G \) be any graph obtained by the above method. We prove the claim by induction on the order of \( G \). If \( G \) consists of only a cycle then its length is \( i \) and any smallest cycle of \( H \) is isomorphic to \( G \). Assume now that \( G \) contains at least one vertex of degree one and let \( v \) be any such vertex of \( G \). Set \( G' = G \setminus v \). We may assume that \( H \) contains an induced copy of \( G' \). Denote this copy of \( G' \) in \( H \) by the very \( G' \). Let \( u \in G' \) be the neighbor of \( v \) in \( G \). It is enough to show that there exists a vertex in \( H \setminus G' \) adjacent to \( u \) but not adjacent to any vertex of \( G' \). Define two subsets as follows: \( A = \{ a \in V(G') : au \in E(G') \} \), \( B = \{ b \in V(H) \setminus V(G') : bu \in E(H) \} \).

It is clear that \( A \cup B \) is independent. Let \( C = V(G') \setminus A \setminus \{ u \} \). The number of edges between \( B \) and \( C \) is at most \((t - 1)|C|\). We claim that there is a vertex say \( z \in B \) which is not adjacent to any vertex of \( C \), since otherwise there will be at least \(|B|\) edges between \( B \) and \( C \). This leads us to \(|B| \leq (t - 1)|C|\). From other side for the order of \( C \) we have \(|C| \leq (4r + 2)(\max\{r + 1, p^{r+1}\})\). Let \( n_{p,r} = (4r + 2)(\max\{r + 1, p^{r+1}\})\). We have therefore \(|B| \leq (t - 1)(n_{p,r} - |A| - 1)\) and \(|A| + |B| \leq (t - 1)n_{p,r} - 1\). But \(|A| + |B| = d(u) > (t - 1)n_{p,r}\), a contradiction. Therefore there is a vertex \( z \) that is adjacent to \( u \) in \( H \) but not adjacent to \( G' \setminus \{ u \} \). By adding the edge \( uz \) to \( G' \) we obtain an induced subgraph of \( H \) isomorphic to \( G \), as desired.

Finally by taking \( c = \max\{g(p, r, t) + 2 , (4r + 2)(t - 1)(\max\{r + 1, p^{r+1}\}) + 1\} \) the proof completes. \( \square \)

Using Proposition 1 and Theorem 1 we obtain the following result.

**Theorem 7** \( \) Fix positive integers \( t \geq 2 \), \( p \) and \( r \). For any \( i = 1, 2, 3, \ldots \), let \( G_i \) be any \( (p, r) \)-unicyclic graph whose cycle has length \( 2i + 2 \). Then \( \text{Forb}(K_{2,t}, G_1, G_2, \ldots) \) is \((\delta, \chi)\)-bounded.

### 4 Concluding remarks

If a family \( \mathcal{F} \) is both \((\delta, \chi)\)-bounded and \( \chi \)-bounded then it satisfies the following stronger result. For any sequence \( G_1, G_2, \ldots \) with \( G_i \in \mathcal{F} \) if \( \delta(G_i) \to \infty \) then \( \omega(G_i) \to \infty \). Let us call any family satisfying the latter property, \((\delta, \omega)\)-bounded family.

The following result of Rödl (originally unpublished) which was later appeared in Kierstead and Rödl ([7] Theorem 2.3) proves the weaker form of Conjecture 1.
Theorem 8  For every fixed tree $T$ and fixed integer $\ell$, and any sequence $G_i \in \text{Forb}(T, K_{\ell, \ell})$, $\chi(G_i) \to \infty$ implies $\omega(G_i) \to \infty$.

Combination of Theorem 3 with Theorem 8 shows that $\text{Forb}(T, K_{\ell, \ell})$ is $(\delta, \omega)$-bounded.

As we noted before the class of even-hole-free graphs is $(\delta, \chi)$-bounded. It was proved in [1] that if $G$ is even-hole-free graph then $\chi(G) \leq 2\omega(G) + 1$. This implies that $\text{Forb}(C_4, C_6, \ldots)$ too is $(\delta, \omega)$-bounded.

References


