Functional Analysis, BSM, Spring 2012

Exercise sheet: norms and bounded operators

Let X be a vector space over $F = \mathbb{R}$ or $\mathbb{C}$; $\|\cdot\|$ is a *norm* if it satisfies the following properties:

- $\|x\| \geq 0$ for any $x \in X$;
- $\|x\| = 0 \Leftrightarrow x = 0$;
- $\|\alpha x\| = |\alpha| \cdot \|x\|$ for any $\alpha \in F$ and $x \in X$;
- $\|x + y\| \leq \|x\| + \|y\|$ for any $x, y \in X$.

$(X, \|\cdot\|)$ is called a *normed space*; $\|x\|$ is the *norm* (or *length*) of the vector $x \in X$.

A normed space $(X, \|\cdot\|)$ is a metric space with the metric $d(x, y) = \|x - y\|$. So we can use all the notions that we defined for metric spaces. A sequence $x_1, x_2, \ldots \in X$ converges to $x \in X$ if $\|x_n - x\| \to 0$ as $n \to \infty$ (that is, for any $\varepsilon > 0$ there exists an N such that $\|x_n - x\| < \varepsilon$ for $n \geq N$). A sequence $x_1, x_2, \ldots \in X$ is Cauchy if for any $\varepsilon > 0$ there exists an N such that $\|x_m - x_n\| < \varepsilon$ for $m, n \geq N$. A normed space is complete if every Cauchy sequence converges. Complete normed spaces are called *Banach spaces*.

Given two normed spaces $(X_1, \|\cdot\|_1)$; $(X_2, \|\cdot\|_2)$, a map $T : X_1 \to X_2$ is a *bounded operator* if

- it is linear, i.e., $T(x + y) = Tx + Ty$ and $T(\alpha x) = \alpha Tx$;
- and it is bounded, i.e., there exists $C \geq 0$ such that $\|Tx\|_2 \leq C\|x\|_1$ for any $x \in X_1$.

The norm (or *operator norm*) of T is defined as the smallest such C, or equivalently:

$$\|T\| = \|T\|_{1,2} \overset{\text{def}}{=} \sup_{x \neq 0} \frac{\|Tx\|_2}{\|x\|_1} = \sup_{\|x\|_1 = 1} \|Tx\|_2 = \sup_{\|x\|_1 \leq 1} \|Tx\|_2.$$

It follows that $\|Tx\|_2 \leq \|T\| \cdot \|x\|_1$ for any $x \in X_1$. By $B(X_1, X_2)$ we denote the space of bounded operators from X_1 to X_2. It is easy to see that $B(X_1, X_2)$ is a normed space with the operator norm. We proved that if X_2 is complete, then so is $B(X_1, X_2)$.

The ℓ_p spaces are important examples of Banach spaces:

$$p = \infty : \ \ell_\infty = \{(\alpha_1, \alpha_2, \ldots) : \alpha_n \in \mathbb{C} \text{ and } (\alpha_n) \text{ is bounded}\} \text{ with the norm } \|(\alpha_1, \alpha_2, \ldots)\|_\infty = \sup_n |\alpha_n|;$$

$$1 \leq p < \infty : \ \ell_p = \left\{(\alpha_1, \alpha_2, \ldots) : \alpha_i \in \mathbb{C} \text{ and } \sum_{i=1}^{\infty} |\alpha_i|^p < \infty\right\} \text{ with the norm } \|(\alpha_1, \alpha_2, \ldots)\|_p = \sqrt[p]{\sum_{i=1}^{\infty} |\alpha_i|^p}.$$

1. Suppose that $x_n \to x$ and $y_n \to y$ in a normed space. Prove that $x_n + y_n \to x + y$.

2. Prove that $x_n \to x$ implies $\|x_n\| \to \|x\|$.

3. Let $C^\infty[0, 1]$ be the space of infinitely differentiable $[0, 1] \to \mathbb{R}$ functions with the sup norm:

$$\|f\| = \sup_{x \in [0,1]} |f(x)|.$$

Is the derivative operator $(f \mapsto f')$ on $C^\infty[0, 1]$ bounded?

4. Consider the left shift operator $T : \ell_1 \to \ell_1$:

$$(\alpha_1, \alpha_2, \alpha_3, \ldots) \mapsto (\alpha_2, \alpha_3, \alpha_4, \ldots).$$

Is T bounded? What is the norm of T?

5. W2P2. (4 points) Consider the following operator:

$$T : (\alpha_1, \alpha_2, \alpha_3, \ldots) \mapsto (\alpha_1, \alpha_1, \alpha_2, \alpha_3, \ldots).$$

This can be viewed as an $\ell_p \to \ell_p$ operator for any $1 \leq p \leq \infty$. Determine the norm of T for each p.

6. a) Consider the space

$$X = \{(\alpha_1, \alpha_2, \ldots) : \alpha_i \in \mathbb{C} \text{ and } \alpha_i = 0 \text{ for all but finitely many } i\text{'s}\}$$

with the ℓ_∞-norm:

$$\|(\alpha_1, \alpha_2, \ldots)\|_\infty = \sup_i |\alpha_i|.$$

We define $T : X \to X$ as

$$(\alpha_1, \alpha_2, \alpha_3, \ldots) \mapsto (\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \alpha_3 - \alpha_4, \ldots).$$

Is T bounded? Determine $\|T\|$. Show that $\ker T = \{0\}$. Prove that T is a bijection from X onto itself.
b) Let S be the following $X \to X$ operator:

$$(\alpha_1, \alpha_2, \alpha_3, \ldots) \mapsto (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \ldots, \alpha_2 + \alpha_3 + \alpha_4 + \ldots, \alpha_3 + \alpha_4 + \ldots, \ldots).$$

(Note that only finitely many α_i are nonzero, so these infinite sums are, in fact, finite sums.) Is S bounded? Determine $\|S\|$. What is the connection between S and T?

7. W2P3. (7 points) For $T \in B(X, X)$ the *spectral radius* of T is

$$\varrho(T) \stackrel{\text{def}}{=} \inf_k \sqrt[k]{\|T^k\|} = \lim_{k\to\infty} \sqrt[k]{\|T^k\|}.$$

What is the spectral radius of the following $\ell_1 \to \ell_1$ operator?

$$(\alpha_1, \alpha_2, \alpha_3, \ldots) \mapsto (2\alpha_1, \alpha_1, \alpha_2, \alpha_3, \ldots)$$

Show an eigenvalue λ for this operator such that $|\lambda|$ is equal to the the spectral radius.

8.* W2P4. (12 points) What is the spectral radius of the following $\ell_\infty \to \ell_\infty$ operator?

$$(\alpha_1, \alpha_2, \alpha_3, \ldots) \mapsto (\alpha_1 + \alpha_2, \alpha_1, \alpha_2, \alpha_3, \ldots)$$

Show an eigenvalue λ for this operator such that $|\lambda|$ is equal to the the spectral radius.

9. Let λ be an eigenvalue for the operator $T \in B(X, X)$. Prove that $|\lambda| \le \|T\|$. Also prove that $|\lambda| \le \varrho(T)$.

10. W2P5. (10 points) Let $(X, \|\cdot\|)$ be a normed space. For a sequence $x_1, x_2, \ldots \in X$ let $s_n = \sum_{i=1}^n x_i$. If (s_n) is convergent, then we call the sequence (x_n) *summable*; we can think of the limit point of (s_n) as the infinite sum $\sum_{i=1}^\infty x_i$. We call the sequence (x_n) *absolutely summable* if $\sum_{i=1}^\infty \|x_i\| < \infty$.

Prove that in a Banach space every absolutely summable sequence is summable.

11.* W2P6. (15 points) Suppose that in a normed space $(X, \|\cdot\|)$ every absolutely summable sequence is summable. Prove that $(X, \|\cdot\|)$ is a Banach space.

Solutions can be found on: `www.renyi.hu/~harangi/bsm/`