Functional Analysis, BSM, Spring 2012

Exercise sheet: bounded linear functionals and dual spaces

Let $(X, \|\cdot\|)$ be a normed space over $F = \mathbb{R}$ or $\mathbb{C}$. A map $\Lambda : X \to F$ is called a *linear functional* if $\Lambda(x+y) = \Lambda x + \Lambda y$ and $\Lambda(\alpha x) = \alpha \Lambda x$ for any $x, y \in X$; $\alpha \in F$. The (operator) norm of Λ is defined as follows:

$$\|\Lambda\| \stackrel{\text{def}}{=} \sup_{x \neq 0} \frac{|\Lambda x|}{\|x\|} = \sup_{\|x\|=1} |\Lambda x|.$$

A linear functional Λ is said to be *bounded* if $\|\Lambda\| < \infty$. The space of all bounded linear functionals on X is called the *dual space* of X and it is denoted by X^*. Equipped with the operator norm, X^* is a Banach space.

Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. We proved that $\ell_p^* = \ell_q$. By that we mean the following: every $\Lambda \in \ell_p^*$ is of the form Λ_y for some $y = (\beta_1, \beta_2, \ldots) \in \ell_q$, where Λ_y is the bounded linear functional on ℓ_p for which

$$\Lambda_y(\alpha_1, \alpha_2, \ldots) = \alpha_1\beta_1 + \alpha_2\beta_2 + \ldots$$

Moreover, the operator norm $\|\Lambda_y\|$ is equal to $\|y\|_q = \sqrt[q]{\sum_{i=1}^{\infty} |\beta_i|^q}$.

1. W3P1. (5 points) Consider the following linear functional on ℓ_3:

$$\Lambda : (\alpha_1, \alpha_2, \alpha_3, \ldots) \mapsto \alpha_1 + \frac{1}{4}\alpha_2 + \frac{1}{4^2}\alpha_3 + \ldots$$

Determine $\|\Lambda\|$.

2. Let Λ be a nonzero bounded linear functional on a real normed space X, and let $y \in X \setminus \ker \Lambda$. Prove that any vector $z \in X$ can be written as $x + \alpha y$ with $x \in \ker \Lambda$ and $\alpha \in \mathbb{R}$.

3. Let $1 < p < \infty$. Prove that for any $\Lambda \in \ell_p^*$ there exists $x \in \ell_p$ with

$$\frac{|\Lambda x|}{\|x\|_p} = \|\Lambda\|.$$

4. Show that the statement of the previous exercise does not hold for $p = 1$: find a bounded linear functional $\Lambda \in \ell_1^*$ such that $\frac{|\Lambda x|}{\|x\|_1} < \|\Lambda\|$ for all nonzero $x \in \ell_1$.

5. Prove that $\ell_1^* = \ell_\infty$.

(You have to prove that for any $y \in \ell_\infty$, Λ_y is a bounded linear functional on ℓ_1 with $\|\Lambda_y\| = \|y\|_\infty$. You also have to prove that any $\Lambda \in \ell_1^*$ is equal to Λ_y for some $y \in \ell_\infty$.)

6. W3P2. (5 points) Let us consider the space

$$c_0 \stackrel{\text{def}}{=} \left\{ (\alpha_1, \alpha_2, \ldots) : \alpha_n \in \mathbb{C} \text{ and } \lim_{n\to\infty} \alpha_n = 0 \right\} \text{ with the } \ell_\infty\text{-norm } \|(\alpha_1, \alpha_2, \ldots)\|_\infty = \sup_n |\alpha_n|.$$

Show that c_0 is separable.

7. W3P3. (5 points) Find a bounded linear functional $\Lambda \in c_0^*$ such that $\frac{|\Lambda x|}{\|x\|_\infty} < \|\Lambda\|$ for all nonzero $x \in c_0$.

8. W3P4. (10 points) Prove that $c_0^* = \ell_1$.

(You have to prove that for any $y \in \ell_1$, Λ_y is a bounded linear functional on c_0 with $\|\Lambda_y\| = \|y\|_1$. You also have to prove that any $\Lambda \in c_0^*$ is equal to Λ_y for some $y \in \ell_1$.)

9. W3P5. (5 points) Let X be a real normed space and suppose that for some $\Lambda_1, \Lambda_2 \in X^*$ we have $\ker \Lambda_1 \subset \ker \Lambda_2$. Prove that $\Lambda_2 = \lambda \Lambda_1$ for some $\lambda \in \mathbb{R}$.

10. Let $C[0,1]$ be the space of continuous $[0,1] \to \mathbb{R}$ functions with the sup norm:

$$\|f\| = \sup_{x \in [0,1]} |f(x)|.$$

Construct as many bounded linear functionals on this space as you can. Determine their operator norms.

11.* W3P6. (12 points) Let $1 < p < \infty$. Use the Hölder inequality to prove that $\|x+y\|_p \le \|x\|_p + \|y\|_p$ for any $x, y \in \ell_p$.

Solutions can be found on: `www.renyi.hu/~harangi/bsm/`