Functional Analysis, BSM, Spring 2012

Exercise sheet: bounded linear functionals and dual spaces

Solutions

1. With $p = 3$ and $q = 3/2$ we have $1/p + 1/q = 1$. Let

$$y = (1, \frac{1}{4}, \frac{1}{4^2}, \frac{1}{4^3}, \ldots).$$

We need to determine the operator norm of $\Lambda_y \in \ell_3^*$:

$$\|\Lambda_y\| = \|y\|_{\frac{3}{2}} = \left(\sum_{n=0}^{\infty} \left(\frac{1}{4^n} \right)^{\frac{3}{2}} \right)^{\frac{2}{3}} = \left(\sum_{n=0}^{\infty} \frac{1}{8^n} \right)^{\frac{2}{3}} = \left(\frac{1}{1 - \frac{1}{8}} \right)^{\frac{2}{3}} = \frac{4}{\sqrt[3]{7^2}}.$$

2. Suppose that z has the desired decomposition $z = x + \alpha y$. Using $x \in \ker \Lambda$ we get that

$$\Lambda z = \Lambda x + \Lambda(\alpha y) = 0 + \alpha \Lambda y = \alpha \Lambda y.$$

Note that $\Lambda y \neq 0$, because $y \notin \ker \Lambda$. Thus α must be $\Lambda z / \Lambda y$, and x must be $z - \alpha y$.

Now pick an arbitrary $z \in X$. Let $\alpha = \Lambda z / \Lambda y$ and $x = z - \alpha y$. All we have to prove is that $x \in \ker \Lambda$, which is clear, since

$$\Lambda x = \Lambda z - \alpha \Lambda y = \Lambda z - \frac{\Lambda z}{\Lambda y} \Lambda y = 0.$$

3. We actually proved this when we proved $\ell_p^* = \ell_q$. Any $\Lambda \in \ell_p^*$ is equal to Λ_y for some $y = (\beta_1, \beta_2, \ldots) \in \ell_q$. We also know that

$$\|\Lambda_y\| = \|y\|_q = \sqrt[q]{\sum_{i=1}^{\infty} |\beta_i|^q}.$$

In the proof we used the following vector $x \in \ell_p$:

$$x = \left(|\beta_1|^{q-2} \bar{\beta}_1, |\beta_2|^{q-2} \bar{\beta}_2, \ldots \right).$$

For this particular x we obtained that

$$\|x\|_p = (\|y\|_q)^{\frac{q}{p}}$$

and

$$|\Lambda_y x| = (\|y\|_q)^q$$

. It follows that

$$\frac{|\Lambda_y x|}{\|x\|_p} = (\|y\|_q)^{q - \frac{q}{p}} = \|y\|_q = \|\Lambda_y\|.$$

4. Let

$$y = \left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots \right)$$

and consider the functional $\Lambda_y \in \ell_1^*$. For $x = (\alpha_1, \alpha_2, \ldots) \in \ell_1$:

$$|\Lambda_y x| = \sum_{i=1}^{\infty} \left| \frac{i}{i+1} \alpha_i \right| \leq \sum_{i=1}^{\infty} |\alpha_i| = \|x\|_1.$$

And equality holds only if each α_i is 0, that is, $x = 0$. Thus for any nonzero $x \in \ell_1$ we have

$$\frac{|\Lambda_y x|}{\|x\|_1} < 1. \tag{1}$$

So it remains to prove that the operator norm of Λ_y is 1. It follows from (1) that $\|\Lambda_y\| \leq 1$. And for $e_n = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$ we have

$$\frac{|\Lambda_y e_n|}{\|e_n\|_1} = \frac{n}{n+1},$$

which tends to 1 as n tends to infinity.

5. First let $y = (\beta_1, \beta_2, \ldots) \in \ell_\infty$ be fixed and let

$$\Lambda_y(\alpha_1, \alpha_2, \ldots) = \alpha_1\beta_1 + \alpha_2\beta_2 + \ldots \tag{2}$$

If $x = (\alpha_1, \alpha_2, \ldots) \in \ell_1$, then

$$\sum_{i=1}^{\infty} |\alpha_i\beta_i| \le \|y\|_\infty \sum_{i=1}^{\infty} |\alpha_i| \le \|y\|_\infty \|x\|_1 < \infty.$$

Thus the sum on the right-hand side of (2) is absolutely convergent, so it is convergent. It means that Λ_y defines a functional on ℓ_1. This functional is clearly linear; it is also bounded, because

$$|\Lambda_y x| = \left|\sum_{i=1}^{\infty} \alpha_i\beta_i\right| \le \sum_{i=1}^{\infty} |\alpha_i\beta_i| \le \|y\|_\infty \|x\|_1.$$

This also implies that $\|\Lambda_y\| \le \|y\|_\infty$. To prove that $\|\Lambda_y\| = \|y\|_\infty$, notice that for $e_n = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$ we have

$$\frac{|\Lambda_y e_n|}{\|e_n\|_1} = |\beta_n|,$$

the supremum of which is, by definition, $\|y\|_\infty$.

It remains to show that any $\Lambda \in \ell_1^*$ is equal to Λ_y for some $y \in \ell_\infty$. Let

$$\beta_n \stackrel{\text{def}}{=} \Lambda e_n,$$

and consider the vector $y = (\beta_1, \beta_2, \ldots)$. We claim that $y \in \ell_\infty$ and $\Lambda = \Lambda_y$. Since $|\beta_n| = |\Lambda e_n| \le \|\Lambda\| \|e_n\|_1 = \|\Lambda\|$, (β_n) is a bounded sequence (with bound $\|\Lambda\|$). Now let x be an arbitrary vector $(\alpha_1, \alpha_2, \ldots)$ in ℓ_1. We need to show that $\Lambda x = \Lambda_y x$. Let x_n be $(\alpha_1, \alpha_2, \ldots, \alpha_n, 0, 0, \ldots)$. It is easy to see that

$$\|x_n - x\|_1 \to 0 \text{ as } n \to \infty.$$

In other words, $x_n \to x$ in ℓ_1. Since both Λ and Λ_y are bounded linear functionals, it follows that $\Lambda x_n \to \Lambda x$ and $\Lambda_y x_n \to \Lambda_y x$. However, $\Lambda x_n = \Lambda_y x_n$ for each n:

$$\Lambda x_n = \sum_{i=1}^{n} \Lambda(\alpha_i e_i) = \sum_{i=1}^{n} \alpha_i \Lambda e_i = \sum_{i=1}^{n} \alpha_i\beta_i = \Lambda_y x_n.$$

Thus Λx must be equal to $\Lambda_y x$, too. We are done.

6. Let S be the set consisting of the points

$$(a_1 + b_1 i, a_2 + b_2 i, \ldots, a_k + b_k i, 0, 0, 0, \ldots),$$

where k is a positive integer, and a_j, b_j ; $1 \le j \le k$ are rational numbers. This set is clearly countable. We claim that it is dense in c_0. We need to show that for any $x = (\alpha_1, \alpha_2, \ldots) \in c_0$ and any $\varepsilon > 0$ there exists a point x' of the above form with $\|x - x'\|_\infty \le \varepsilon$. Since $\alpha_n \to \infty$, there exists N such that $|\alpha_n| < \varepsilon$ for $n \ge N$. Let $k = N$ and for any $j \le N$ let us choose rational numbers a_j, b_j such that $|\alpha_j - (a_j + b_j i)| < \varepsilon$. It is clear that

$$x' = (a_1 + b_1 i, a_2 + b_2 i, \ldots, a_N + b_N i, 0, 0, 0, \ldots)$$

has the desired property.

7. Let $y = (1/2, 1/4, 1/8, \ldots)$ and consider the linear functional Λ_y. For $x = (\alpha_1, \alpha_2, \ldots)$:

$$|\Lambda_y x| = \left|\frac{1}{2}\alpha_1 + \frac{1}{4}\alpha_2 + \ldots\right| \le (1/2 + 1/4 + \ldots)\|x\|_\infty = \|x\|_\infty.$$

Equality holds if and only if $\alpha_1 = \alpha_2 = \ldots$. If $x \in c_0 \setminus \{0\}$, then this cannot be the case. Thus for any nonzero $x \in c_0$ we have

$$\frac{|\Lambda_y x|}{\|x\|_\infty} < 1.$$

So it remains to show that $\|\Lambda_y\| = 1$. We have already seen that $\|\Lambda_y\| \le 1$. For $s_n = (1, 1, \ldots, 1, 0, 0, \ldots) \in c_0$:

$$\frac{|\Lambda_y s_n|}{\|s_n\|_\infty} = \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n},$$

which tends to 1 as $n \to \infty$.

8. First let $y = (\beta_1, \beta_2, \ldots) \in \ell_1$ be fixed and let

$$\Lambda_y(\alpha_1, \alpha_2, \ldots) = \alpha_1\beta_1 + \alpha_2\beta_2 + \ldots \tag{3}$$

If $x = (\alpha_1, \alpha_2, \ldots) \in c_0$, then

$$\sum_{i=1}^{\infty} |\alpha_i\beta_i| \le \|x\|_\infty \sum_{i=1}^{\infty} |\beta_i| \le \|x\|_\infty \|y\|_1 < \infty.$$

Thus the sum on the right-hand side of (3) is absolutely convergent, so it is convergent. It means that Λ_y defines a functional on c_0. This functional is clearly linear; it is also bounded, because

$$|\Lambda_y x| = \left|\sum_{i=1}^{\infty} \alpha_i\beta_i\right| \le \sum_{i=1}^{\infty} |\alpha_i\beta_i| \le \|x\|_\infty \|y\|_1.$$

This also implies that $\|\Lambda_y\| \le \|y\|_1$. To prove that $\|\Lambda_y\| = \|y\|_1$, consider the following vector for each n:

$$s_n = \left(\frac{\bar{\beta}_1}{|\beta_1|}, \frac{\bar{\beta}_2}{|\beta_2|}, \ldots, \frac{\bar{\beta}_n}{|\beta_n|}, 0, 0, \ldots\right).$$

Notice that $\bar{\beta}_i/|\beta_i|$ are all complex numbers with unit length (if $\beta_i = 0$, then pick any complex number of unit length). So $\|s_n\|_\infty = 1$.

$$\frac{|\Lambda_y s_n|}{\|s_n\|_\infty} = \sum_{i=1}^{n} \frac{\beta_i\bar{\beta}_i}{|\beta_i|} = \sum_{i=1}^{n} \frac{|\beta_i|^2}{|\beta_i|} = \sum_{i=1}^{n} |\beta_i|,$$

which tends to $\|y\|_1$ as $n \to \infty$.

It remains to show that any $\Lambda \in c_0^*$ is equal to Λ_y for some $y \in \ell_1$. Let

$$\beta_n \stackrel{\text{def}}{=} \Lambda e_n,$$

and consider the vector $y = (\beta_1, \beta_2, \ldots)$. We claim that $y \in \ell_1$. Let s_n be defined as above. Then

$$\Lambda s_n = \sum_{i=1}^{n} |\beta_i|.$$

Since $\|s_n\|_\infty = 1$, we have $\sum_{i=1}^{n} |\beta_i| \le \|\Lambda\|$ for each n. It follows that $\sum_{i=1}^{\infty} |\beta_i| \le \|\Lambda\|$. Thus $y \in \ell_1$ as we wanted. It means that Λ_y is a bounded linear functional on c_0; we claim that $\Lambda = \Lambda_y$. Let x be an arbitrary vector $(\alpha_1, \alpha_2, \ldots)$ in ℓ_1, and let x_n be $(\alpha_1, \alpha_2, \ldots, \alpha_n, 0, 0, \ldots)$. Since $x \in c_0$, $\alpha_n \to 0$, thus

$$\|x_n - x\|_\infty \to 0 \text{ as } n \to \infty.$$

In other words, $x_n \to x$ in c_0. Since both Λ and Λ_y are bounded linear functionals on c_0, it follows that $\Lambda x_n \to \Lambda x$ and $\Lambda_y x_n \to \Lambda_y x$. However, $\Lambda x_n = \Lambda_y x_n$ for each n:

$$\Lambda x_n = \sum_{i=1}^{n} \Lambda(\alpha_i e_i) = \sum_{i=1}^{n} \alpha_i \Lambda e_i = \sum_{i=1}^{n} \alpha_i\beta_i = \Lambda_y x_n.$$

Thus Λx must be equal to $\Lambda_y x$, too. We are done.

9. If $\ker \Lambda_1$ is the whole space X, then so is $\ker \Lambda_2$, thus $\Lambda_1 = \Lambda_2 = 0$, we are done.

If $\ker \Lambda_1 \ne X$, then pick an arbitrary $y \in X \setminus \ker \Lambda_1$. Let $\lambda_1 = \Lambda_1 y$ and $\lambda_2 = \Lambda_2 y$. We claim that $\Lambda_2 = \lambda \Lambda_1$ with $\lambda = \lambda_2/\lambda_1$. Exercise 2 says that any $z \in X$ can be written as $x + \alpha y$, where $\alpha \in \mathbb{R}$ and $x \in \ker \Lambda_1$. Notice that then x is also in $\ker \Lambda_2$, so $\Lambda_1 x = \Lambda_2 x = 0$. It follows that

$$\Lambda_2 z = \Lambda_2 x + \alpha \Lambda_2(y) = 0 + \alpha\lambda_2 = \lambda\left(0 + \alpha\lambda_1\right) = \lambda\left(\Lambda_1 x + \alpha\Lambda_1(y)\right) = \lambda\Lambda_1 z.$$

10. Pick a finite number of points in the interval: $x_1, \dots, x_k \in [0,1]$; and let $\alpha_1, \dots, \alpha_k$ be arbitrary real numbers. Then

$$\Lambda f = \sum_{i=1}^{k} \alpha_i f(x_i)$$

defines a bounded linear operator on $C[0,1]$ with $\|\Lambda\| = \sum_{i=1}^{n} |\alpha_i|$.

Another bounded linear functional:

$$\Lambda f = \int_0^1 f(x)\,\mathrm{d}x,$$

the operator norm of which is 1.

A generalization of the previous example: fix a continuous function $k : [0,1] \to \mathbb{R}$ and let

$$\Lambda f = \int_0^1 f(x)k(x)\,\mathrm{d}x.$$

The norm of this functional is $\int_0^1 |k(x)|\,\mathrm{d}x$.

A common generalization of all the above examples is the following. Let $g : [0,1] \to \mathbb{R}$ be a function of bounded variation; consider the Riemann-Stieltjes integral with respect to g:

$$\Lambda f = \int_0^1 f(x)\,\mathrm{d}g(x).$$

The norm of Λ is the total variation $V(g)$ of g. (Actually, it can be shown that all bounded linear functionals on $C[0,1]$ are of this form.)

11.* Let us replace each coordinate of x and y with its absolute value. It clearly suffices to prove the statement for these vectors. Thus we can assume that in each coordinate we have nonnegative real numbers.

We have seen earlier that if x, y are in ℓ_p, then so is $x+y$. Let q be such that $1/p + 1/q = 1$. The key observation is the following: since $(p-1)q = p$, the vector $(x+y)^{p-1}$ is in ℓ_q, its ℓ_q-norm is $\|x+y\|_p^{p-1}$. We apply the Hölder inequality to $x \in \ell_p$ and $(x+y)^{p-1} \in \ell_q$ as well as to $y \in \ell_p$ and $(x+y)^{p-1} \in \ell_q$:

$$\|x(x+y)^{p-1}\|_1 \le \|x\|_p \|x+y\|_p^{p-1}$$

and

$$\|y(x+y)^{p-1}\|_1 \le \|y\|_p \|x+y\|_p^{p-1}.$$

It follows that

$$\|x+y\|_p^p = \|(x+y)^p\|_1 \le \|x(x+y)^{p-1}\|_1 + \|y(x+y)^{p-1}\|_1 \le (\|x\|_p + \|y\|_p)\,\|x+y\|_p^{p-1},$$

which implies that $\|x+y\|_p \le \|x\|_p + \|y\|_p$.