Functional Analysis, BSM, Spring 2012

Exercise sheet: extension of functionals and the Hahn-Banach theorem

Let $(X, \|\cdot\|)$ be a (real or complex) normed space and let $Y \leq X$ be a linear subspace. Note that Y is also a normed space with $\|\cdot\|$. The *Hahn-Banach theorem* states that for any $\Lambda \in Y^*$ there exists $\widetilde{\Lambda} \in X^*$ such that $\widetilde{\Lambda}$ is an extension of Λ ($\widetilde{\Lambda} y = \Lambda y$ for any $y \in Y$) and $\|\widetilde{\Lambda}\| = \|\Lambda\|$.

1. Let x, y be distinct vectors in a normed space X. Use the Hahn-Banach theorem to show that there exists $\Lambda \in X^*$ for which $\Lambda x \neq \Lambda y$.

2. W3P7. (5 points) Use the Hahn-Banach theorem to prove that for any nonzero element x_0 of a normed space X there exists $\Lambda \in X^*$ such that $\Lambda(x_0) = \|x_0\|$ and $\|\Lambda\| = 1$.

3. W3P8. (5 points) Consider the following linear subspace of the Banach space ℓ_∞:

$$M = \{(\alpha_1, \alpha_2, \ldots) : \alpha_i \in \mathbb{C} \text{ and } \alpha_i = 0 \text{ for all but finitely many } i\text{'s}\}.$$

Show that M is a linear subspace of ℓ_∞. Is it closed? What is its closure?

4. Let M be as in the previous exercise. Use the Hahn-Banach theorem to prove the existence of a bounded linear functional $\Lambda \in \ell_\infty^*$ for which $\Lambda x = 0$ for any $x \in M$ and

$$\Lambda(1, 1, 1, \ldots) = 1.$$

5. Prove that $\ell_\infty^* \neq \ell_1$. In other words, find $\Lambda \in \ell_\infty^*$ such that $\Lambda \neq \Lambda_y$ for any $y \in \ell_1$.

6. Prove the next theorem by adapting the proof of the original Hahn-Banach theorem.

The generalized Hahn-Banach theorem. Let V be a **real vector space** and let $p : V \to \mathbb{R}$ be a sublinear functional, that is,

$$p(x + y) \leq p(x) + p(y) \text{ for any } x, y \in V \text{ and } p(\alpha x) = \alpha p(x) \text{ for any } x \in V \text{ and } \textbf{positive} \text{ real } \alpha.$$

Let $M \leq V$ be a linear subspace with a linear functional $\Lambda : M \to \mathbb{R}$ such that

$$\Lambda y \leq p(y) \text{ for any } y \in M.$$

Then Λ can be extended to a linear functional $\widetilde{\Lambda} : V \to \mathbb{R}$ such that

$$\widetilde{\Lambda} y = \Lambda y \text{ for any } y \in M \text{ and } \widetilde{\Lambda} x \leq p(x) \text{ for any } x \in V.$$

7. Show that the *real case* of the original Hahn-Banach theorem follows from the generalized Hahn-Banach theorem.

8. W3P9. (12 points) Let ℓ_∞ denote this time the **real** vector space of bounded sequences of real numbers. Use the generalized Hahn-Banach theorem to prove the existence of a linear functional $\Lambda : \ell_\infty \to \mathbb{R}$ such that for any $x = (\alpha_1, \alpha_2, \ldots) \in \ell_\infty$ it holds that

$$\liminf_{n\to\infty} \alpha_n \leq \Lambda x \leq \limsup_{n\to\infty} \alpha_n.$$

Show that such a linear functional is necessarily bounded (with respect to the ℓ_∞ norm). Also show that such a functional is different from any Λ_y; $y \in \ell_1$.

9. W3P10. (12 points) Let $(X, \|\cdot\|)$ be a Banach space and let Y be a closed linear subspace, $Y \neq X$. Prove that for any $\varepsilon > 0$ there exists $e \in X$ such that $\|e\| = 1$ and $d(e, Y) > 1 - \varepsilon$, where $d(e, Y)$ denotes the distance of e and Y, that is, $\inf_{y \in Y} \|e - y\|$.

Solutions can be found on: `www.renyi.hu/~harangi/bsm/`