Functional Analysis, BSM, Spring 2012

Exercise sheet: extension of functionals and the Hahn-Banach theorem

Solutions

1. Let X be over the field F ($\mathbb{R}$ or $\mathbb{C}$). Let $z = x - y$. We need to find a functional $\Lambda \in X^*$ for which $\Lambda z = \Lambda x - \Lambda y \neq 0$. We can easily define such a bounded linear functional on the one-dimensional linear subspace spanned by z, $Y = \{\alpha z : \alpha \in F\}$. Namely, let

$$\Lambda(\alpha z) = \alpha.$$

It is easy to see that Λ is a linear functional on Y with $\Lambda z = 1 \neq 0$ and $\|\Lambda\| = 1/\|z\|$. By Hahn-Banach theorem there exists a linear functional $\widetilde{\Lambda}$ on X with the same properties.

2. Consider the linear subspace spanned by x_0: $Y = \{\alpha x_0 : \alpha \in F\}$. We define a linear functional Λ on Y by

$$\Lambda(\alpha x_0) = \alpha \|x_0\|.$$

It is clear that $\Lambda x_0 = \|x_0\|$ and $\|\Lambda\| = 1$. By Hahn-Banach theorem there exists a linear functional $\widetilde{\Lambda}$ on X with the same properties.

3. Clearly, $M \subset \ell_\infty$. To see that M is a linear subspace, we have to check two things: $x + y \in M$ for any $x, y \in M$ and $\alpha x \in M$ for any $\alpha \in \mathbb{C}, x \in M$. Both are clear in this case.

The set M is not a closed set in ℓ_∞, since the vectors

$$x_n = (1, 1/2, 1/3, \ldots, 1/n, 0, 0, \ldots)$$

are all in M, but they converge (in ℓ_∞-norm) to

$$x = (1, 1/2, 1/3, 1/4, \ldots),$$

which is not in M.

We claim that he closure of M, $\mathrm{cl}(M)$ is

$$c_0 = \left\{ (\alpha_1, \alpha_2, \ldots) : \alpha_n \in \mathbb{C} \text{ and } \lim_{n \to \infty} \alpha_n = 0 \right\}.$$

On the one hand, we need to show that for any $x \in c_0$ there exists a sequence (x_n) in M such that x_n converges to x (in the ℓ_∞-norm). Pick any $x = (\alpha_1, \alpha_2, \ldots) \in c_0$. The vectors

$$x_n = (\alpha_1, \alpha_2, \ldots, \alpha_n, 0, 0, \ldots)$$

are all in M and

$$\|x - x_n\|_\infty = \|(0, 0, \ldots, 0, \alpha_{n+1}, \alpha_{n+2}, \ldots)\|_\infty = \sup_{i \geq n+1} |\alpha_i|,$$

which goes to 0 as $n \to \infty$, because $\alpha_n \to 0$.

On the other hand, we need that for any convergent sequence (x_n) with $x_n \in M$, the limit point x lies in c_0. Let $x = (\alpha_1, \alpha_2, \ldots)$; we have to show that $\alpha_i \to 0$ as $i \to \infty$. Let $\varepsilon > 0$ be arbitrary. There exists n_0 such that

$$\|x - x_{n_0}\|_\infty < \varepsilon.$$

However, x_{n_0} is in M, so there exists N such that after the N-th coordinate x_{n_0} has only zeros. Since the ℓ_∞-distance of x and x_{n_0} is less than ε, it follows that for $i > N$ we have $|\alpha_i| < \varepsilon$. We are done.

4. Set $e = (1, 1, \ldots)$ and let Y be the linear subspace spanned by M and e:

$$Y = \{x + \alpha e : x \in M; \alpha \in F\}.$$

First we define Λ on this subspace:

$$\Lambda(x + \alpha e) = \alpha.$$

This is clearly a linear functional on Y. Is it bounded? We claim that

$$\frac{|\Lambda(x + \alpha e)|}{\|x + \alpha e\|_\infty} = \frac{|\alpha|}{\|x + \alpha e\|_\infty} \leq 1,$$

which would imply that $\|\Lambda\| \leq 1$. (In fact, $\|\Lambda\| = 1$.) We need that $\|x + \alpha e\|_\infty \geq |\alpha|$. We know that $x \in M$, so it is 0 in all but finitely many coordinates. It means that $x + \alpha e$ is α in all but finitely many coordinates. So the ℓ_∞-norm of $x + \alpha e$ is at least $|\alpha|$, and this is what we wanted to prove.

So Λ is a bounded linear functional on Y with the desired properties. By Hahn-Banach theorem we can extend it to a bounded linear functional on ℓ_∞.

5. We claim that the bounded linear operator $\Lambda \in \ell_\infty^*$ of the previous exercise has the property that $\Lambda \neq \Lambda_y$ for any $y \in \ell_1$. We prove this by contradiction. Assume that $\Lambda = \Lambda_y$ for some $y = (\beta_1, \beta_2, \ldots) \in \ell_1$. Let $e_n = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$. Note that $e_n \in M$ for all n. So

$$\Lambda e_n = 0.$$

On the other hand,

$$\Lambda e_n = \Lambda_y e_n = \beta_n.$$

We conclude that $\beta_n = 0$ for all n. In other words, $y = 0$. Thus $\Lambda = \Lambda_0$ is the constant zero functional, but this contradicts $\Lambda(1, 1, \ldots) = 1$.

6. Let $z \in V \setminus M$. We prove that Λ can be extended to a linear functional Λ_1 on

$$M_1 = \{y + \alpha z : y \in M; \alpha \in \mathbb{R}\}.$$

The rest of the proof (the Zorn's lemma argument) goes the same way as in the original theorem.

If Λ_1 extends Λ, then

$$\Lambda_1(y + \alpha z) = \Lambda y + \alpha \Lambda_1 z.$$

So the value of $\Lambda_1 z$ determines Λ_1. We need to choose $\Lambda_1 z$ such that

$$\Lambda y + \alpha \Lambda_1 z \leq p(y + \alpha z)$$

for any $y \in M$ and $\alpha \in \mathbb{R}$.

If $\alpha = 0$, then we get back our assumption $\Lambda y \leq p(y)$.

If $\alpha > 0$, then after dividing by α and rearranging we get that

$$\Lambda_1 z \leq p\left(\frac{y}{\alpha} + z\right) - \Lambda\left(\frac{y}{\alpha}\right).$$

If $\alpha > 0$, then after dividing by $-\alpha$ and rearranging we get that

$$\Lambda_1 z \geq -p\left(\frac{y}{-\alpha} - z\right) + \Lambda\left(\frac{y}{-\alpha}\right).$$

(We needed to distinguish these cases, because $p(\alpha x) = \alpha p(x)$ holds only for positive α.)

So we need to choose $\Lambda_1 z$ in such a way that

$$\inf_{y' \in M} p(y' + z) - \Lambda y' \geq \Lambda_1 z \geq \sup_{y'' \in M} -p(y'' - z) + \Lambda y''.$$

We can do that if and only if for any $y', y'' \in M$ it holds that

$$p(y' + z) - \Lambda y' \geq -p(y'' - z) + \Lambda y'' \Leftrightarrow p(y' + z) + p(y'' - z) \geq \Lambda(y' + y'').$$

Using the sublinearity of p and the assumption that $p(y) \geq \Lambda y$ for $y \in M$:

$$p(y' + z) + p(y'' - z) \geq p((y' + z) + (y'' - z)) = p(y' + y'') \geq \Lambda(y' + y'').$$

7. Let $\Lambda \in Y^*$. We can assume without loss of generality that $\|\Lambda\| = 1$. We will use the generalized Hahn-Banach theorem with $p(x) \stackrel{\text{def}}{=} \|x\|$. This is clearly a sublinear functional (it satisfies both $p(x+y) \le p(x)+p(y)$ and $p(\alpha x) = \alpha p(x)$ for positive α). Moreover, $\Lambda y \le |\Lambda y| \le \|\Lambda\|\|y\| = \|y\|$, so $\Lambda y \le p(y)$ for any $y \in Y$. Thus all the conditions of the generalized Hahn-Banach theorem are satisfied; there exists a linear functional $\widetilde{\Lambda}$ on X such that $\widetilde{\Lambda}$ extends Λ and $\widetilde{\Lambda} x \le p(x) = \|x\|$. Since $\widetilde{\Lambda}$ is an extension, it is clear that $\|\widetilde{\Lambda}\| \ge \|\Lambda\|$. We need to show $\|\widetilde{\Lambda}\| \le \|\Lambda\| = 1$, that is,

$$|\widetilde{\Lambda} x| \le \|x\|. \tag{1}$$

We already know that

$$\widetilde{\Lambda} x \le \|x\|. \tag{2}$$

Using this for $-x$ instead of x we get that

$$-\widetilde{\Lambda} x = \widetilde{\Lambda}(-x) \le \|-x\| = \|x\|.$$

It follows that

$$\widetilde{\Lambda} x \ge -\|x\|.$$

Combining this with (2) we get (1).

8. It is easy to see that

$$p(\alpha_1, \alpha_2, \ldots) \stackrel{\text{def}}{=} \limsup_{n\to\infty} \alpha_n$$

is a sublinear functional on ℓ_∞. We use the generalized Hahn-Banach theorem in the special case when $M = 0$. In this special case the theorem only states that there exists some linear functional Λ on ℓ_∞ with

$$\Lambda(\alpha_1, \alpha_2, \ldots) \le \limsup_{n\to\infty} \alpha_n. \tag{3}$$

We claim that such a Λ necessarily satisfies the other desired inequality

$$\Lambda(\alpha_1, \alpha_2, \ldots) \ge \liminf_{n\to\infty} \alpha_n. \tag{4}$$

We just need to use (3) for the vector $(-\alpha_1, -\alpha_2, \ldots)$:

$$-\Lambda(\alpha_1, \alpha_2, \ldots) = \Lambda(-\alpha_1, -\alpha_2, \ldots) \le \limsup_{n\to\infty} -\alpha_n = -\liminf_{n\to\infty} \alpha_n,$$

which implies (4).

To show that Λ is bounded, we notice that

$$\limsup_{n\to\infty} \alpha_n \le \|(\alpha_1, \alpha_2, \ldots)\|_\infty \text{ and } \liminf_{n\to\infty} \alpha_n \ge -\|(\alpha_1, \alpha_2, \ldots)\|_\infty.$$

It follows that $|\Lambda x| \le \|x\|$ for any $x \in \ell_\infty$, thus $\|\Lambda\| \le 1$.

Note that such a bounded linear functional Λ has the properties that $\Lambda y = 0$ for $y \in M$ and $\Lambda(1, 1, \ldots) = 1$ (see Exercise 4). So the same argument as in Exercise 5 shows that Λ is different from any Λ_y; $y \in \ell_1$.

9. Pick an arbitrary vector x outside Y. Since Y is closed, $d(x, Y) > 0$ (otherwise there would be a sequence $y_n \in Y$ with $\|y_n - x\| \to 0$, but Y is closed, so this would mean that x also lies in Y).

Let $\delta > 0$. Since $d(x, Y)$ is the infimum of $\|x - y\|$, there exists $y_0 \in Y$ such that

$$\|x - y_0\| \le (1 + \delta) d(x, Y).$$

Let

$$e \stackrel{\text{def}}{=} \frac{x - y_0}{\|x - y_0\|}.$$

Clearly, $\|e\| = 1$. For the distance of e and and an arbitrary $y \in Y$ we have

$$\|e - y\| = \left\| \frac{x - y_0}{\|x - y_0\|} - y \right\| = \frac{1}{\|x - y_0\|} \left\| x - \underbrace{y_0 - \|x - y_0\| \cdot y}_{\in Y} \right\| \ge \frac{d(x, Y)}{\|x - y_0\|} \ge \frac{d(x, Y)}{(1 + \delta) d(x, Y)} = \frac{1}{1 + \delta},$$

which is at least $1 - \varepsilon$, if we choose δ small enough.