Functional Analysis, BSM, Spring 2012

Exercise sheet: Spectra of operators

Let X be a complex Banach space and let $T \in B(X) = B(X, X)$, that is, $T : X \to X$ is a bounded operator. T is *invertible* if it has a bounded inverse $T^{-1} \in B(X)$. By the inverse mapping theorem T is invertible if and only if it is bijective, that is, T is injective ($\ker T = \{0\}$) and surjective ($\operatorname{ran} T = X$).
The *resolvent set* of T is

$$\varrho(T) \stackrel{\text{def}}{=} \{\lambda \in \mathbb{C} : \lambda I - T \text{ is invertible}\} = \{\lambda : \lambda I - T \text{ is bijective}\}.$$

The *spectrum* of T is

$$\sigma(T) \stackrel{\text{def}}{=} \mathbb{C} \setminus \varrho(T) = \{\lambda : \lambda I - T \text{ is not invertible}\} = \{\lambda : \lambda I - T \text{ is not bijective}\}.$$

The *point spectrum* of T (the set of eigenvalues) is

$$\sigma_p(T) \stackrel{\text{def}}{=} \{\lambda : \lambda I - T \text{ is not injective}\} = \{\lambda : \exists x \in X \setminus \{0\} \text{ such that } Tx = \lambda x\}.$$

The *residual spectrum* of T is

$$\sigma_r(T) \stackrel{\text{def}}{=} \{\lambda : \lambda I - T \text{ is injective and } \operatorname{ran}(\lambda I - T) \text{ is not dense}\}.$$

It can be shown that the spectrum $\sigma(T)$ is a non-empty closed set for which

$$\sigma(T) \subset \{\lambda \in \mathbb{C} : |\lambda| \le \|T\|\}, \text{ moreover, } \sigma(T) \subset \{\lambda \in \mathbb{C} : |\lambda| \le r(T)\},$$

where $r(T)$ denotes the spectral radius of T: $r(T) \stackrel{\text{def}}{=} \inf_k \sqrt[k]{\|T^k\|}$.
If T is invertible, then $\sigma(T^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(T)\}$.
If $p(z) = \sum_{i=0}^m \alpha_i z^i$ is a polynomial, then for $p(T) \stackrel{\text{def}}{=} \sum_{i=0}^m \alpha_i T^i \in B(X)$ we have $\sigma(p(T)) = \{p(\lambda) : \lambda \in \sigma(T)\}$.

1. Let X, Y be normed spaces. Prove that $\ker T$ is a closed linear subspace of X for any bounded linear operator $T : X \to Y$. (In particular, $\ker \Lambda$ is a closed linear subspace of X for any $\Lambda \in X^*$.)

2. a) Let X, Y be normed spaces, $T : X \to Y$ a bounded operator. Show that the range $\operatorname{ran} T = \{Tx : x \in X\}$ is a linear subspace of Y.
b) Let $X = Y = \ell_2$ and let $T : \ell_2 \to \ell_2$ be the following operator:

$$T(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots) = (\alpha_1, \alpha_2/2, \alpha_3/3, \alpha_4/4, \ldots).$$

Show that T is bounded and $\operatorname{ran} T$ is not closed.

3. W5P1. (8 points) Let X, Y be Banach spaces. Suppose that $T \in B(X, Y)$ is *bounded below*, that is, there exists $c > 0$ such that $\|Tx\| \ge c\|x\|$ for all $x \in X$. Prove that $\operatorname{ran} T$ is closed.

4. W5P2. (8 points) Let X be a Banach space, $T \in B(X)$. Show that T is invertible if and only if T is bounded below and $\operatorname{ran} T$ is dense.

5. Let T be the left shift operator on ℓ_∞.
a) Determine $\|T\|$ and the point spectrum $\sigma_p(T)$.
b) What is the spectrum and the residual spectrum of T?

6. Let T be the left shift operator on ℓ_1.
a) Determine $\|T\|$, $\sigma_p(T)$, $\sigma(T)$.
b) Prove that $\operatorname{ran}(I - T)$ is dense by showing that it contains all sequences with finitely many nonzero elements.
c) Show that $\operatorname{ran}(I - T) \ne \ell_1$. Can you give an explicit example of a vector $y \notin \operatorname{ran}(I - T)$?
d) What is the residual spectrum of T?

7. Let $T \in B(X)$. Prove that $\operatorname{ran} T$ is not dense if and only if there exists a nonzero $\Lambda \in X^*$ such that $\Lambda T = 0$.

8. W5P3. (8 points) Let T be the right shift operator on ℓ_1.
a) Determine $\|T\|$ and $\sigma_p(T)$.
b) Prove that if $|\lambda| \le 1$, then $\operatorname{ran}(\lambda I - T)$ is not dense. (Hint: find $0 \ne \Lambda \in \ell_1^*$ such that $\Lambda(\lambda I - T) = 0$.)
c) Determine $\sigma_r(T)$ and $\sigma(T)$.

9. W5P4. (10 points) Let T be the right shift operator on ℓ_∞.
a) Find an open ball in ℓ_∞ that is disjoint from $\mathrm{ran}(I-T)$.
b) Determine $\sigma_p(T)$, $\sigma_r(T)$ and $\sigma(T)$.

10. W5P5. (12 points) Let T be the right shift operator on ℓ_2.
a) Show that $\mathrm{ran}(I-T) \neq \ell_2$, but $\mathrm{ran}(I-T)$ is dense in ℓ_2.
b) Determine $\sigma_p(T)$, $\sigma_r(T)$ and $\sigma(T)$.

11. W5P6. (10 points) Consider the space $C[0,1]$ with the supremum norm. The *Volterra integral operator* $T: C[0,1] \to C[0,1]$ is defined as

$$(Tf)(x) \stackrel{\text{def}}{=} \int_0^x f(y)\,\mathrm{d}y.$$

Determine the norm and the spectral radius of T. What is $\sigma(T)$ and $\ker T$? Is $\mathrm{ran}\, T$ closed in $C[0,1]$?

12. Let X be a normed space, $T \in B(X)$. Prove that $\lim_{k\to\infty} \sqrt[k]{\|T^k\|}$ exists and is equal to the spectral radius $r(T) = \inf_k \sqrt[k]{\|T^k\|}$.
Hint: show that the sequence $a_k = \log\|T^k\|$ is *subadditive* (i.e., $a_{m+n} \leq a_m + a_n$) and prove that for any such sequence $\lim_{k\to\infty} a_k/k$ exists and is equal to $\inf_k a_k/k$.

13. Let $S, T \in B(X)$ be invertible bounded operators. Prove that

$$\text{if } \|S-T\| \leq \frac{1}{2\|T^{-1}\|}, \text{ then } \left\|S^{-1}\right\| \leq 2\left\|T^{-1}\right\|.$$

14. Consider the following operator on ℓ_1:

$$T: (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots) \mapsto (\alpha_1+\alpha_2+\alpha_3, \alpha_2+\alpha_3+\alpha_4, \alpha_3+\alpha_4+\alpha_5, \alpha_4+\alpha_5+\alpha_6, \ldots).$$

Describe $\sigma(T)$. What is its intersection with the real axis?

15. W5P7. (8 points) Let X be a Banach space, $S, T \in B(X)$. Show that $r(ST) = r(TS)$.

16. Let X be a Banach space, $T \in B(X)$ with $r(T) < 1$. Prove that $I + T + T^2 + T^3 + \cdots$ is a convergent sum in $B(X)$. Show that the limit operator is the inverse of $I - T$.

17.* W5P8. (20 points) Let X be a Banach space, $S, T \in B(X)$.
a) Prove that $I - ST$ is invertible if and only if $I - TS$ is invertible.
b) Prove that $\{0\} \cup \sigma(ST) = \{0\} \cup \sigma(TS)$. In other words, the spectra of ST and TS are the same with the possible exception of the point 0.

Solutions can be found on: `www.renyi.hu/~harangi/bsm/`