Functional Analysis, BSM, Spring 2012

Exercise sheet: Hilbert spaces

The closest point theorem: Let H be a Hilbert space, $K \subset H$ a convex, closed subset. Then for any $x_0 \in H$ there exists a unique $y_0 \in K$ such that

$$\|x_0 - y_0\| = d(x_0, K).$$

(In most applications K is a closed linear subspace.)
Riesz-lemma: Let H be a Hilbert space, $H_1 \leq H$ a closed linear subspace. Then

$$H = H_1 \oplus H_1^{\perp}, \text{ i.e., } \forall x \in H \ \exists! x_1 \in H_1 \ \exists! x_2 \in H_1^{\perp} \text{ s.t. } x = x_1 + x_2.$$

(We say that x_1 is the *orthogonal projection* of x onto the closed linear subspace H_1; we write $x_1 = P_{H_1} x$.)
Riesz representation theorem: Let H be a Hilbert space. For any $\Lambda \in H^*$ there exists a unique $y \in H$ such that

$$\Lambda x = \Lambda_y x \stackrel{\text{def}}{=} (x, y) \text{ for all } x \in H.$$

The mapping $y \mapsto \Lambda_y$ is a bijection from H to H^* with the following properties:

- isometric: $\|y\| = \|\Lambda_y\|$;
- conjugate linear: $\Lambda_{y_1+y_2} = \Lambda_{y_1} + \Lambda_{y_2}$ and $\Lambda_{\alpha y} = \overline{\alpha}\Lambda_y$.

1. Prove the Pythagorean theorem for Hilbert spaces: if $x \perp y$, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

2. Find a Banach space X, a closed linear subspace $Y \leq X$ and a vector $x_0 \in X$ such that the closest point of Y to x_0 is not unique.

3. W9P5. (8 points) Let H be a Hilbert space.
a) If M is a closed linear subspace of H, then $\left(M^{\perp}\right)^{\perp} = M$.
b) If M is an arbitrary subset of H, then $\left(M^{\perp}\right)^{\perp} = \operatorname{cl}(\operatorname{span} M)$.

4. W9P6. (12 points) Hahn-Banach theorem says that if X is a normed space and $Y \leq X$ is a linear subspace, then any bounded linear functional $\Lambda \in Y^*$ can be extended to a bounded linear functional $\widetilde{\Lambda} \in X^*$ such that $\|\Lambda\| = \|\widetilde{\Lambda}\|$. Show that this extension is unique if X is a Hilbert space.

5. Let M be a finite dimensional subspace of a Hilbert space H. Suppose that $e_1, \ldots, e_n$ is an orthonormal basis for M (that is, $\{e_1, \ldots, e_n\}$ is a basis for M and $(e_i, e_i) = 1$, $(e_i, e_j) = 0$ for any $i \neq j$). Prove that

$$\sum_{i=1}^{n} (x, e_i) e_i \in M$$

is the closest point of M to x.

Solutions can be found on: `www.renyi.hu/~harangi/bsm/`