Functional Analysis, BSM, Spring 2012

Exercise sheet: Orthonormal bases

Solutions

1. a) Let

$$x_n = \sum_{i=1}^{n} \alpha_i s_i.$$

We need to prove that the sequence (x_n) is convergent. Since the space is complete, it suffices to show that (x_n) is Cauchy. Our assumption $\sum_{i=1}^{\infty} |\alpha_i|^2 < \infty$ implies that for every $\varepsilon > 0$ there exists N such that $\sum_{i=N+1}^{\infty} |\alpha_i|^2 < \varepsilon$. For any $n > m \geq N$ we have

$$\|x_n - x_m\|^2 = \left\| \sum_{i=m+1}^{n} \alpha_i s_i \right\|^2 = \sum_{i=m+1}^{n} |\alpha_i|^2 \leq \sum_{i=m+1}^{\infty} |\alpha_i|^2 < \varepsilon.$$

b)

$$(x_n, y_n) = \left(\sum_{i=1}^{n} \alpha_i s_i, \sum_{j=1}^{n} \beta_j s_j \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\beta_j} (s_i, s_j) = \sum_{i=1}^{n} \alpha_i \overline{\beta_i}.$$

Since $x_n \to x$ and $y_n \to y$, we have $(x_n, y_n) \to (x, y)$, thus

$$(x, y) = \sum_{i=1}^{\infty} \alpha_i \overline{\beta_i}.$$

2. $(i) \Rightarrow (ii)$: Let $M = \{x_1, x_2, \ldots\}$ be a countable dense set in H and let S be an arbitrary orthonormal system. Since M is dense, for each $s \in S$ we can pick $x_i \in M$ such that $\|s - x_i\| < 1/2$. However, the same x_i cannot be picked for two distinct s: if $s, s' \in S$ and $s \neq s'$, then

$$\|s - s'\|^2 = \|s\|^2 + \|s'\|^2 = 2, \text{ so } \|s - s'\| = \sqrt{2}.$$

So different $s \in S$ get different $x_i \in M$. Since M is countable, so is S.
$(ii) \Rightarrow (iii)$: Trivial if we use the fact that every Hilbert space has an orthonormal basis.
$(iii) \Rightarrow (i)$: Let $\{s_1, s_2, \ldots\}$ be an orthonormal basis of H. We claim that the following countable set is dense in H

$$M = \left\{ \sum_{i=1}^{k} \beta_i s_i : k \in \mathbb{N}; \Re\beta_i, \Im\beta_i \in \mathbb{Q} \right\}.$$

By the Theorem each vector $x \in H$ can be written in the form

$$x = \sum_{i=1}^{\infty} \alpha_i s_i, \text{ where } \sum_{i=1}^{\infty} |\alpha_i|^2 < \infty.$$

Fix a positive ε and choose $k \geq 2$ such that $\sum_{i=k+1}^{\infty} |\alpha_i|^2 < \varepsilon^2/2$. Then for each $1 \leq i \leq k$ pick $\beta_i \in \mathbb{C}$ with $\Re\beta_i, \Im\beta_i \in \mathbb{Q}$ and with $|\alpha_i - \beta_i| < \varepsilon/k$. Setting $y = \sum_{i=1}^{k} \beta_i s_i \in M$ we get

$$\|x - y\|^2 = \left\| \sum_{i=1}^{k} (\alpha_i - \beta_i) s_i + \sum_{i=k+1}^{\infty} \alpha_i s_i \right\|^2 = \sum_{i=1}^{k} |\alpha_i - \beta_i|^2 + \sum_{i=k+1}^{\infty} |\alpha_i|^2 < k \left(\frac{\varepsilon}{k} \right)^2 + \frac{\varepsilon^2}{2} \leq \varepsilon^2.$$

3. $(i) \Rightarrow (ii)$: Assume that there exists $x \neq 0$ with $x \perp s$ $(\forall s \in S)$. Then $x/\|x\|$ can be added to the orthonormal system S, so it cannot be complete.
$(ii) \Rightarrow (i)$: Assume that S is not complete. Then there exists s' such that $S \cup \{s'\}$ is an orthonormal system. This means that $s' \perp s$ $(\forall s \in S)$, which implies $s' = 0$ by (ii), contradiction.
$(ii) \Leftrightarrow (iii)$:

$$(ii) \Leftrightarrow S^{\perp} = \{0\} \Leftrightarrow \left(S^{\perp}\right)^{\perp} = H \Leftrightarrow \mathrm{cl}(\mathrm{span}\, S) = H \Leftrightarrow (iii).$$

$(i) \Rightarrow (iv)$: This is the statement of the Theorem.
$(iv) \Rightarrow (v)$:

$$(x,y) = \left(\sum_{s\in S} (x,s)s, \sum_{t\in S} (y,t)t \right) = \sum_{s\in S}\sum_{t\in S} (x,s)\overline{(y,t)}(s,t) = \sum_{s\in S} (x,s)\overline{(y,s)}.$$

The above computation is certainly correct in the case when only finitely many terms are nonzero in both sums. The case when there are are (countably) infinitely many nonzero terms follows from the finite case and the continuity of the inner product.
$(v) \Rightarrow (vi)$: Plug $y = x$ and use $(x,s)(s,x) = (x,s)\overline{(x,s)} = |(x,s)|^2$.
$(vi) \Rightarrow (i)$: Assume that S is not complete. Then there exists s' such that $S \cup \{s'\}$ is an orthonormal system. Using (vi) with $x = s'$:

$$\|s'\|^2 = \sum_{s\in S} |(s',s)|^2 = 0,$$

contradicting $\|s'\| = 1$.

4.* If H is finite dimensional (as a vector space), then the claim follows from the linear algebraic theorem saying that two bases of the same vector space have the same cardinality. So we may assume that H is infinite dimensional. Then S and T are clearly infinite sets. We have seen that for any fixed $x \in H$ the set

$$S_x \overset{\text{def}}{=} \{s \in S : (x,s) \neq 0\}$$

is countable. We claim that

$$S = \bigcup_{t\in T} S_t. \tag{1}$$

We need to show that for any $s \in S$ there exists $t \in T$ such that $s \in S_t$, that is, $(t,s) \neq 0$. Assume that $s \perp t$ for all $t \in T$. Since T is a complete orthonormal system, this would imply that $s = 0$, leading to a contradiction. So we proved (1); since each S_t is countable, it follows that $|S| \leq |T|\aleph_0 = |T|$. Changing the roles of S and T we get $|T| \leq |S|$, too.

5. a) It is easy to see that the L_2-norm of each function in S is 1. Since $\int_0^1 \Psi_{n,k}\,\mathrm{d}\lambda = 0$ for all n,k, it follows that the constant 1 function is orthogonal to all Haar functions $\Psi_{n,k}$. It remains to show that $\Psi_{n,k} \perp \Psi_{n',k'}$ if $(n,k) \neq (n',k')$. If $n = n'$ (and $k \neq k'$), then they have disjoint support and the claim is trivial. If $n \neq n'$ (say $n > n'$), then $\Psi_{n',k'}$ is always constant on $\operatorname{supp}\Psi_{n,k}$ and the claim follows from $\int_0^1 \Psi_{n,k}\,\mathrm{d}\lambda = 0$.
b) Since f is orthogonal to the constant 1 function, we have $\int_0^1 f\,\mathrm{d}\lambda = 0$, which proves the statement for $x = 1$. Now we claim that

$$\int_{k2^{-n}}^{(k+1)2^{-n}} f\,\mathrm{d}\lambda = 0 \text{ for any } n \in \mathbb{N} \text{ and } 0 \leq k \leq 2^n - 1.$$

We prove this by induction: if it is true for a given pair (n,k), then using that $f \perp 2^{-n/2}\Psi_{n,k}$ we get that it is also true for the pairs $(n+1, 2k)$ and $(n+1, 2k+1)$. It follows that $\int_0^x f\,\mathrm{d}\lambda = 0$ for any $x = k2^{-n}$. Since these dyadic rationals are dense in $[0,1]$, for any $x \in [0,1]$ there exists a sequence (x_n) of dyadic rationals such that $x_n \to x$. Then

$$0 = \int_0^{x_n} f\,\mathrm{d}\lambda = \int_0^1 f\chi_{[0,x_n]}\,\mathrm{d}\lambda \to \int_0^1 f\chi_{[0,x]}\,\mathrm{d}\lambda = \int_0^x f\,\mathrm{d}\lambda.$$

We used Lebesgue's dominated convergence theorem for the functions $f\chi_{[0,x_n]}$ (they are all dominated by $|f| \in L_2[0,1] \subset L_1[0,1]$.
c) We have seen that if $S^\perp = \{0\}$ for an orthonormal system S, then S is complete. We proved in part b) that for any $f \in S^\perp$ we have $\int_0^x f\,\mathrm{d}\lambda = 0$ for all $x \in [0,1]$. It remains to show that this implies that $f = 0$ almost everywhere. We wish to show that

$$\int_E f\,\mathrm{d}\lambda = 0 \text{ for all measurable sets } E \subset [0,1].$$

We know this for $E = [0,x]$, which easily yields that it is true when E is the finite union of intervals, and every Lebesgue measurable set can be approximated by such sets.

If we use this for the measurable sets

$$E_+ = \{x : f(x) > 0\} \text{ and } E_- = \{x : f(x) < 0\},$$

then we get $\lambda(E_+) = \lambda(E_-) = 0$, thus $f = 0$ λ-almost everywhere and the proof is complete. If f is complex-valued, then we need to consider the sets

$$\{x : \Re f(x) > 0\}\,; \{x : \Re f(x) < 0\}\,; \{x : \Im f(x) > 0\}\,; \{x : \Im f(x) < 0\}\,.$$

6.* First we construct a curve $\gamma : [0,1] \to L_2[0,1]$ with the desired properties. Let

$$\gamma(t) = \chi_{[0,t)},$$

the characteristic function of the interval $[0,t)$. Then $\gamma(b) - \gamma(a) = \chi_{[a,b)}$. Clearly, $\chi_{[a,b)} \perp \chi_{[c,d)}$ if $0 \le a < b \le c < d \le 1$. Also, γ is continuous since

$$\|\gamma(b) - \gamma(a)\| = \|\chi_{[a,b)}\| = \sqrt{b-a}.$$

Now let H be an arbitrary infinite dimensional Hilbert space. There exists a countable orthonormal system $S = \{s_1, s_2, \ldots\}$ in H. Let $M = \mathrm{cl}(\operatorname{span} S)$ be the closure of the subspace spanned by $s_1, s_2, \ldots$; then $M \cong L_2[0,1]$, because both have dimension $\aleph_0$. So there exists a curve $\gamma : [0,1] \to M \subset H$ with the desired properties; we are done.