Functional Analysis, BSM, Spring 2012
Exercise sheet: Operators on Hilbert spaces
Solutions

Proof of the listed properties of the Hilbert adjoint.

$$((S+T)x,y) = (Sx+Tx,y) = (Sx,y)+(Tx,y) = (x,S^*y)+(x,T^*y) = (x,(S^*+T^*)y), \text{ so } (S+T)^* = S^*+T^*;$$
$$((\alpha T)x,y) = (\alpha Tx,y) = \alpha(Tx,y) = \alpha(x,T^*y) = (x,(\overline{\alpha}T^*)y), \text{ so } (\alpha T)^* = \overline{\alpha}T^*;$$
$$(STx,y) = (S(Tx),y) = (Tx,S^*y) = (x,T^*(S^*y)) = (x,(T^*S^*)y), \text{ so } (ST)^* = T^*S^*.$$

Now we show that if T has a bounded inverse T^{-1} (i.e., $TT^{-1} = T^{-1}T = I$), then T^* is also invertible and

$$(T^*)^{-1} = \left(T^{-1}\right)^*.$$

It is easy to see from the definition that $I^* = I$, so using that $(ST)^* = T^*S^*$ we obtain

$$I = I^* = \left(T^{-1}T\right)^* = T^*\left(T^{-1}\right)^* \text{ and } I = I^* = \left(TT^{-1}\right)^* = \left(T^{-1}\right)^* T^*,$$

so $\left(T^{-1}\right)^*$ is the inverse of T^* as claimed.

$$(T^*x,y) = \overline{(y,T^*x)} = \overline{(Ty,x)} = (x,Ty), \text{ thus } (T^*)^* = T.$$

We proved in class that $\|T^*\| \leq \|T\|$. Therefore

$$\|T\| = \left\|(T^*)^*\right\| \leq \|T^*\| \leq \|T\|.$$

Finally, we prove $\|T^*T\| = \|T\|^2$. We know that $\|ST\| \leq \|S\|\|T\|$, so using that $\|T^*\| = \|T\|$ we get

$$\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2.$$

For the reverse inequality, we notice that by Cauchy inequality

$$\|Tx\|^2 = (Tx,Tx) = (x,T^*Tx) \leq \|x\|\|T^*Tx\| \leq \|x\|\|T^*T\|\|x\| = \|T^*T\|\|x\|^2.$$

Consequently,

$$\|T\|^2 = \left(\sup_{x\neq 0} \frac{\|Tx\|}{\|x\|}\right)^2 = \sup_{x\neq 0} \frac{\|Tx\|^2}{\|x\|^2} \leq \|T^*T\|.$$

1. a) Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{C}^n$. Then

$$M_{i,j} = (Te_i, e_j).$$

Let M^* denote the matrix corresponding to T^*. We have

$$M^*_{i,j} = (T^*e_i,e_j) = \overline{(e_j,T^*e_i)} = \overline{(Te_j,e_i)} = \overline{M_{j,i}},$$

so M^* is the conjugate transpose of M.
b) T is self-adjoint if and only if $M_{i,j} = \overline{M_{j,i}}$ holds for all i,j.
2. Let L denote the left shift operator. Then $T = \frac{1}{2}(I+L)$. It is easy to see that the adjoint of the left shift operator is the right shift operator R, thus $T^* = \frac{1}{2}(I+R)$.
3. Using $(ST)^* = T^*S^*$ and $(T^*)^* = T$:

$$(TT^*)^* = (T^*)^* T^* = TT^*.$$

Similarly,

$$(T^*T)^* = T^*(T^*)^* = T^*T.$$

4. If T is self-adjoint, then for any $x \in H$

$$(Tx, x) = (x, T^*x) = (x, Tx) = \overline{(Tx, x)},$$

therefore (Tx, x) is real.

For the other direction, let $x, y \in H$ and $\alpha \in \mathbb{C}$. Then

$$(T(\alpha x + y), \alpha x + y) = |\alpha|^2 (Tx, x) + (Ty, y) + \alpha(Tx, y) + \overline{\alpha}(Ty, x) \in \mathbb{R}.$$

Since the first two terms on the right are also real,

$$\alpha(Tx, y) + \overline{\alpha}(Ty, x) = \alpha(Tx, y) + \overline{\alpha(x, Ty)} = \alpha(Tx, y) - \alpha(x, Ty) + \alpha(x, Ty) + \overline{\alpha(x, Ty)} = \\ \alpha(Tx, y) - \alpha(x, Ty) + 2\Re\left(\alpha(x, Ty)\right) \in \mathbb{R}.$$

It follows that $\alpha(Tx, y) - \alpha(x, Ty)$ is real for all $\alpha \in \mathbb{C}$. Therefore $(Tx, y) = (x, Ty)$; $T = T^*$.

5.* Let

$$C = \sup_{\|x\|=1} |(Tx, x)| = \sup_{x \neq 0} \frac{|(Tx, x)|}{\|x\|^2}.$$

If $\|x\| = 1$, then it follows from Cauchy inequality that

$$|(Tx, x)| \le \|Tx\|\|x\| \le \|T\|\|x\|^2 = \|T\|, \text{ so } C \le \|T\|.$$

To show the reverse inequality, suppose that $\|x\| = \|y\| = 1$. Using

$$(T(x + y), x + y) - (T(x - y), x - y)) = 2(Tx, y) + 2(Ty, x) = 4\Re(Tx, y)$$

and the parallelogram law we obtain

$$4\Re(Tx, y) \le C\left(\|x + y\|^2 + \|x - y\|^2\right) = 2C\left(\|x\|^2 + \|y\|^2\right) = 4C.$$

Let α be a scalar with $|\alpha| = 1$ such that $|(Tx, y)| = \alpha(Tx, y) = (T(\alpha x), y)$. Then

$$|(Tx, y)| = \Re(T(\alpha x), y) \le C.$$

Plugging $y = Tx/\|Tx\|$ we get

$$\|Tx\| = \left|\left(Tx, \frac{Tx}{\|Tx\|}\right)\right| \le C, \text{ whenever } \|x\| = 1.$$

This implies $\|T\| \le C$, we are done.

6. a) If $(Tx, x) = 0$ for all $x \in H$, then by the previous exercise

$$\|T\| = \sup_{\|x\|=1} |(Tx, x)| = 0, \text{ thus } T = 0.$$

(If H is a complex Hilbert space, then $(Tx, x) = 0$ implies that T is self-adjoint, so we can omit that assumption. For a real Hilbert space H of dimension at least 2, however, there exists a nonzero $T \in B(H)$ such that $(Tx, x) = 0$ for all $x \in H$.)

b) Since

$$(Tx, Tx) = (T^*Tx, x) \text{ and } (T^*x, T^*x) = (TT^*x, x),$$

it follows that $\|Tx\| = \|T^*x\|$ holds for all $x \in H$ if and only if

$$((T^*T - TT^*)x, x) = 0 \text{ for all } x \in H,$$

which is equivalent with $T^*T - TT^* = 0$ by part a), because $T^*T - TT^*$ is self-adjoint (since both T^*T and TT^* are self-adjoint).

7.* We prove by contradiction. Assume that T is not bounded, that is, there exist $y_1, y_2, \ldots \in H$ such that

$$\|y_n\| \le 1 \text{ for all } n \text{ and } \|Ty_n\| \to \infty \text{ as } n \to \infty.$$

Consider the following linear functionals on H:

$$\Lambda_n x \stackrel{\text{def}}{=} (Sx, y_n) = (x, Ty_n).$$

Riesz representation theorem tells us that Λ_n is bounded with $\|\Lambda_n\| = \|Ty_n\| \to \infty$. Then the uniform boundedness principle yields that there exists $x \in H$ for which the sequence $|\Lambda_n x|$ is unbounded. However,

$$|\Lambda_n x| = |(Sx, y_n)| \le \|Sx\|\|y_n\| \le \|Sx\|,$$

contradiction. So T is bounded.

Since $(Sx, y) = (x, Ty)$ implies that $(y, Sx) = (Ty, x)$, the same proof yields that S is bounded, too.