Functional Analysis, BSM, Spring 2012
Exercise sheet: Normal operators
Solutions

1. We claim that the adjoint of the left shift operator $L : \ell_2 \to \ell_2$ is the right shift operator $R : \ell_2 \to \ell_2$. Let $x = (\alpha_1, \alpha_2, \ldots)$ and $y = (\beta_1, \beta_2, \ldots)$. Then

$$(Lx, y) = ((\alpha_2, \alpha_3, \alpha_4, \ldots), (\beta_1, \beta_2, \beta_3, \ldots)) = \alpha_2\overline{\beta_1} + \alpha_3\overline{\beta_2} + \alpha_4\overline{\beta_3} + \cdots$$

and

$$(x, Ry) = ((\alpha_1, \alpha_2, \alpha_3, \ldots), (0, \beta_1, \beta_2, \ldots)) = 0 + \alpha_2\overline{\beta_1} + \alpha_3\overline{\beta_2} + \alpha_4\overline{\beta_3} + \cdots ,$$

therefore $L^* = R$. Since

$$LRx = (\alpha_1, \alpha_2, \alpha_3, \ldots) \text{ and } RLx = (0, \alpha_2, \alpha_3, \ldots),$$

it follows that neither L nor R is normal.

2. a)

$$x \in \ker T^* \Leftrightarrow T^*x = 0 \Leftrightarrow (T^*x, y) = 0 \text{ for all } y \in H \Leftrightarrow (x, Ty) = 0 \text{ for all } y \in H \Leftrightarrow$$
$$(x, z) = 0 \text{ for all } z \in \operatorname{ran} T \Leftrightarrow x \in (\operatorname{ran} T)^\perp.$$

b) Plug $T = T^*$ in a) and use $(T^*)^* = T$.

c) Take the orthogonal complement of both sides of a) and use $\big((\operatorname{ran} T)^\perp\big)^\perp = \operatorname{cl}(\operatorname{ran} T)$.

3. For normal operators we have $\|Tx\| = \|T^*x\|$. Thus $Tx = 0 \Leftrightarrow T^*x = 0$.

4. a) Recall that in a Banach space X a bounded operator $T \in B(X)$ is invertible if and only if T is bounded below and $\operatorname{ran} T$ is dense. However, using 2c) we get

$$\operatorname{ran} T \text{ dense } \Leftrightarrow \operatorname{cl}(\operatorname{ran} T) = H \Leftrightarrow (\ker T^*)^\perp = H \Leftrightarrow \ker T^* = \{0\}.$$

b) For a normal operator $\ker T^* = \ker T$, so we get that T is invertible if and only if it is bounded below and $\ker T = \{0\}$. However, if T is bounded below, then $\ker T = \{0\}$ holds automatically.

5. Using $(S + T)^* = S^* + T^*$, $(\alpha T)^* = \overline{\alpha} T^*$ and $I^* = I$ we get that

$$(\lambda I - T)^* = \overline{\lambda} I - T^*.$$

It easily follows that if T is normal, then so is $\lambda I - T$. Therefore

$$Tx = \lambda x \Leftrightarrow x \in \ker(\lambda I - T) \Leftrightarrow x \in \ker(\lambda I - T)^* = \ker(\overline{\lambda} I - T^*) \Leftrightarrow T^*x = \overline{\lambda} x.$$

6. By the previous exercise $T^*y = \overline{\mu} y$. Thus

$$\lambda(x, y) = (\lambda x, y) = (Tx, y) = (x, T^*y) = (x, \overline{\mu} y) = \mu(x, y).$$

Consequently, $(\lambda - \mu)(x, y) = 0$. Since $\lambda \neq \mu$, it follows that $(x, y) = 0$, that is, $x \perp y$.

7. a) Let $x \in \ker T$, that is, $Tx = 0$. Then

$$TT^*x = T^*Tx = T^*0 = 0, \text{ so } T^*x \in \ker T.$$

b) Actually, $Tx \in (\ker T)^\perp$ for any $x \in H$: for every $y \in \ker T = \ker T^*$ we have

$$(Tx, y) = (x, T^*y) = (x, 0) = 0,$$

therefore $Tx \in (\ker T)^\perp$.

c-d) Let $x \in H$ be arbitrary. There exist $y \in \ker T$ and $z \in (\ker T)^\perp$ such that $x = y + z$. Then

$$Tx = Ty + Tz = Tz, \text{ thus } T^k x = T^k z \text{ for all } k \geq 1.$$

We know from part b) that $(\ker T)^\perp$ is T-invariant. Also, $\ker T \cap (\ker T)^\perp = \{0\}$. So if we restrict T to $(\ker T)^\perp$, then we get an injective $(\ker T)^\perp \to (\ker T)^\perp$ operator. Therefore $z = 0 \Leftrightarrow Tz = 0 \Leftrightarrow T^2 z = 0 \Leftrightarrow T^3 z = 0 \Leftrightarrow \cdots$.